Fermi gas model

Introduction to Nuclear Science

Simon Fraser University
Spring 2011

NUCS 342 — February 2, 2011

Outline

Outline

Outline

1. Bosons and fermions
2. The Fermi gas model
3. Infinitely deep potential well (again!)

Outline

1. Bosons and fermions
2. The Fermi gas model
3. Infinitely deep potential well (again!)
4. Fermi momentum and energy

Outline

Wave function for a two-particle system

- For two non-interacting particles, each with mass m, in a potential well $V(x)$ the Shrödinger equation reads:

$$E\psi(x_1,x_2) = (E_1+E_2)\psi(x_1,x_2) =$$
$$= (K_1+V_1+K_2+V_2)\psi(x_1,x_2) =$$
$$= \left(-\frac{\hbar^2}{2m}\frac{d}{dx_1}+V(x_1)-\frac{\hbar^2}{2m}\frac{d}{dx_2}+V(x_2)\right)\psi(x_1,x_2)$$

- The resulting wave function $\psi(x_1,x_2)$ is a product of wave functions resulting from a solution of the corresponding Shrödinger equation for a single particle of mass m in the well V:

$$\psi(x_1,x_2) = \psi_\alpha(x_1)\psi_\beta(x_2)$$

- For this solution the energy is

$$E = E_\alpha + E_\beta$$

Particle exchange transformation

- Note that if

$$\psi(x_1, x_2) = \psi_\alpha(x_1)\psi_\beta(x_2)$$

is a solution of energy

$$E = E_\alpha + E_\beta$$

then

$$\psi'(x_1, x_2) = \psi_\beta(x_1)\psi_\alpha(x_2)$$

is a solution of energy

$$E' = E_\beta + E_\alpha = E$$

- Wave functions $\psi(x_1, x_2)$ and $\psi'(x_1, x_2)$ are related by the exchange of particles $x_1 \rightleftarrows x_2$

$$\psi'(x_1, x_2) = \psi(x_2, x_1)$$

Indistinguishable particles

- Atoms of the same element, nucleons, electrons, photons, and other elementary particles are indistinguishable.
- This implies that given the energy $E = E'$ we can not truly distinguish between the $\psi(x_1, x_2)$ and the $\psi'(x_1, x_2)$ solution of the Shrödinger equation for a two-particle system.
- The correct description of the system has to incorporate this fact into the resulting wave function.

Bosons and fermions

- Bosons are particles which are described by the wave function which is symmetric with respect to exchange of any two particles

$$\psi(..., x_p, ...x_r...) = \psi(..., x_r, ...x_p...)$$

- Fermions are particles which are described by the wave function which is antisymmetric with respect to exchange of any two particles.

$$\psi(..., x_p, ...x_r...) = -\psi(..., x_r, ...x_p...)$$

A two-boson system

- If two particles in the well $V(x)$ we considered before are bosons we need to combine the $\psi(x_1, x_2)$ and $\psi'(x_1, x_2)$ solutions into a wave function symmetric with respect to particle exchange.
- This solution is

$$\phi(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\psi(x_1, x_2) + \psi'(x_1, x_2)\right) =$$
$$= \frac{1}{\sqrt{2}}\left(\psi(x_1, x_2) + \psi(x_2, x_1)\right) =$$
$$= \frac{1}{\sqrt{2}}\left(\psi_\alpha(x_1)\psi_\beta(x_2) + \psi_\beta(x_1)\psi_\alpha(x_2)\right) =$$
$$= \frac{1}{\sqrt{2}}\left(\psi_\alpha(x_1)\psi_\beta(x_2) + \psi_\alpha(x_2)\psi_\beta(x_1)\right)$$

- Note, that the energy for $\phi(x_1, x_2)$, $\psi(x_1, x_2)$, and $\psi'(x_1, x_2)$ is the same $E = E_\alpha + E_\beta$.

A two-boson system

- The bosonic wave function $\phi(x_1, x_2)$ contains correlations between particles 1 and 2 which are absent in the wave function $\psi(x_1, x_2)$ or $\psi'(x_1, x_2)$.
- To see that let us examine a probability for two particles and two bosons being at the same place $x = x_1 = x_2$:

$$
\begin{aligned}
&\psi(x_1 = x, x_2 = x) = \psi_\alpha(x)\psi_\beta(x) \\
&\psi'(x_1 = x, x_2 = x) = \psi_\beta(x)\psi_\alpha(x) = \psi(x_1 = x, x_2 = x) \\
&\phi(x_1 = x, x_2 = x) = \frac{1}{\sqrt{2}}\left(\psi_\alpha(x)\psi_\beta(x) + \psi_\beta(x)\psi_\alpha(x)\right) = \\
&= \sqrt{2}\psi_\alpha(x)\psi_\beta(x) = \sqrt{2}\psi(x_1 = x, x_2 = x)
\end{aligned}
$$

- For two bosons probability of being at the same place is larger than for independent particles. The symmetrization of the wave function implies attraction between bosons.

A two-fermion system

- If two particles in the well $V(x)$ we considered before are fermions we need to combine the $\psi(x_1, x_2)$ and $\psi'(x_1, x_2)$ solutions into a invention antisymmetric with respect to particle exchange.
- This solution is

$$\chi(x_1, x_2) = \frac{1}{\sqrt{2}} \left(\psi(x_1, x_2) - \psi'(x_1, x_2)\right) =$$

$$= \frac{1}{\sqrt{2}} \left(\psi(x_1, x_2) - \psi(x_2, x_1)\right) =$$

$$= \frac{1}{\sqrt{2}} \left(\psi_\alpha(x_1)\psi_\beta(x_2) - \psi_\beta(x_1)\psi_\alpha(x_2)\right) =$$

$$= \frac{1}{\sqrt{2}} \left(\psi_\alpha(x_1)\psi_\beta(x_2) - \psi_\alpha(x_2)\psi_\beta(x_1)\right)$$

- Note, that the energy for $\chi(x_1, x_2)$, $\psi(x_1, x_2)$, and $\psi'(x_1, x_2)$ is the same $E = E_\alpha + E_\beta$.

A two-fermion system

- The fermionic wave function $\chi(x_1, x_2)$ contains correlations between particles 1 and 2 which are absent in the wave function $\psi(x_1, x_2)$ or $\psi'(x_1, x_2)$.

- To see that let us examine a probability for two particles and two fermions being at the same place $x = x_1 = x_2$:

$$\psi(x_1 = x, x_2 = x) = \psi_\alpha(x)\psi_\beta(x)$$
$$\psi'(x_1 = x, x_2 = x) = \psi_\beta(x)\psi_\alpha(x) = \psi(x_1 = x, x_2 = x)$$
$$\phi(x_1 = x, x_2 = x) = \frac{1}{\sqrt{2}}\left(\psi_\alpha(x)\psi_\beta(x) - \psi_\beta(x)\psi_\alpha(x)\right) = 0$$

- For two fermions probability of being at the same place is zero. The antisymmetrization of the wave function implies repulsion between fermions.

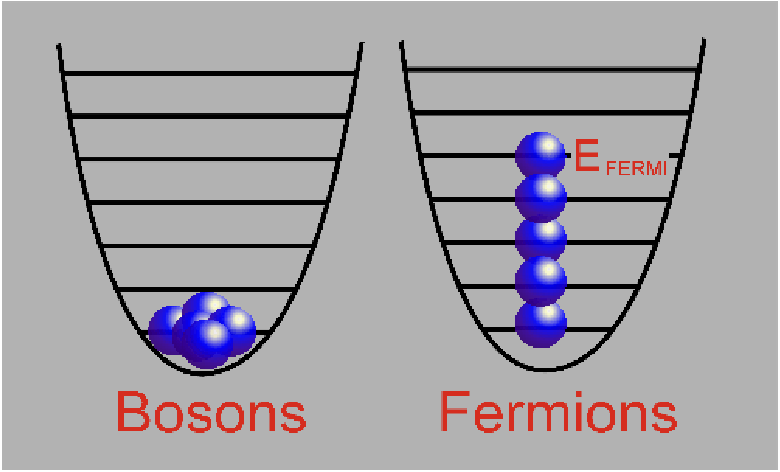
A multi-boson system at zero temperature

- At zero temperature all bosons will occupy the lowest state in the potential well.
- Bosonic attraction will result in a collapse of the system into a coherent state of high density, so called Bose-Einstein condensate.
- Observation of the Bose-Einstein condensate in cold atomic clouds trapped with laser light led to the Nobel Prize in 1997.
- The total energy of bosonic system at zero temperature is equal to the number of bosons times the energy of the lowest state in the well.

A multi-fermion system at zero temperature

- Fermions can not occupy the same state, even at zero temperature.
- At zero temperature the first fermion sets in the lowest state in the potential well, each next fermion sets in the lowest free state in the well.
- Consequently, n fermions occupy the n lowest-energy states in the potential well.
- The energy of the highest occupied state is called the Fermi energy. At zero temperature all states up to the Fermi energy are occupied, while all states above the Fermi energy are free.
- The total energy of the system is equal to the sum of energies of the occupied states.

Bosons and fermions at zero temperature



Spin-statistics theorem

Spin-statistic theorem says that:

- particles with integer spins are bosons,
- particles with half-integer spins are fermions.

For example

- ^{4}He is a boson (true for isotopes with even neutron number),
- ^{3}He is a fermion (true for isotopes with odd neutron number).

The Fermi gas model

- The Fermi gas model defines properties of a system of non-interacting fermions in an infinite potential well.
- The model predicts gross properties of various quantum-mechanical systems, for example electrons in metals, or nucleons in nuclei.
- The model assumes that all fermions occupy the lowest energy states available to them up to the Fermi energy, and that there is no excitations across the Fermi energy (i.e. zero temperature).
- In nuclei the model assumes that protons and neutrons are independent fermion filling two separate potential wells.
- The model assumes, however, common Fermi energy for the protons and neutrons in stable nuclei.
- If Fermi energy for protons and neutrons are different then the β decay transforms one type of nucleons into the other until the common Fermi energy (stability) is reached.

Infinitely deep potential well in three dimensions

- Let us denote the dimensions of the well along the x, y and z coordinates as L_x, L_y and L_z.
- The wave functions are

$$\begin{aligned} \Psi_{n_x,n_y,n_z}(x,y,z) &= \Psi_{n_x}(x)\Psi_{n_y}(y)\Psi_{n_z}(z) = \quad (1) \\ = \quad \sin(k_x x)\sin(k_y y)\sin(k_z z) &= \sin(n_x\pi\frac{x}{L_x})\sin(n_y\pi\frac{y}{L_y})\sin(n_z\pi\frac{z}{L_z}) \end{aligned}$$

- Note that $n_x > 0$, $n_y > 0$ and $n_z > 0$ otherwise $\Psi_{n_x,n_y,n_z} = 0$.
- The energies are

$$\begin{aligned} E_{n_x,n_y,n_z} &= \frac{\pi^2\hbar^2}{2m}\frac{n_x^2}{L_x^2} + \frac{\pi^2\hbar^2}{2m}\frac{n_y^2}{L_y^2} + \frac{\pi^2\hbar^2}{2m}\frac{n_z^2}{L_z^2} = \\ &= \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right) \quad (2) \end{aligned}$$

Infinitely deep potential well in three dimensions

- Recall that the components of the momentum of the particle are

$$p_x = \hbar k_x = \hbar\pi \frac{n_x}{L_x}, \quad p_y = \hbar k_y = \hbar\pi \frac{n_y}{L_y}, \quad p_z = \hbar k_z = \hbar\pi \frac{n_z}{L_z} \quad (3)$$

- The energy is then

$$E_{n_x,n_y,n_z} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right) \quad (4)$$

- And for simplicity we assume (unrealistically) $L_x = L_y = L_z = L$ thus

$$E_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right) \quad (5)$$

Number of states

- Let us denote the Fermi energy equal by E_F.
- The corresponding Fermi momentum is

$$E_F = \frac{p_F^2}{2m} \implies p_F = \sqrt{2mE_F} \tag{6}$$

- Let us count the number of occupied states up to the Fermi momentum. For these states

$$p_x^2 + p_y^2 + p_z^2 < p_F^2 \implies \frac{\pi^2\hbar^2}{L^2}\left(n_x^2 + n_y^2 + n_z^2\right) < p_F^2 \tag{7}$$

- To count the number of states we will use a trick. To use the trick we need to note that Eq. 7 implies

$$n_x^2 + n_y^2 + n_z^2 < \frac{p_F^2 L^2}{\pi^2\hbar^2} \tag{8}$$

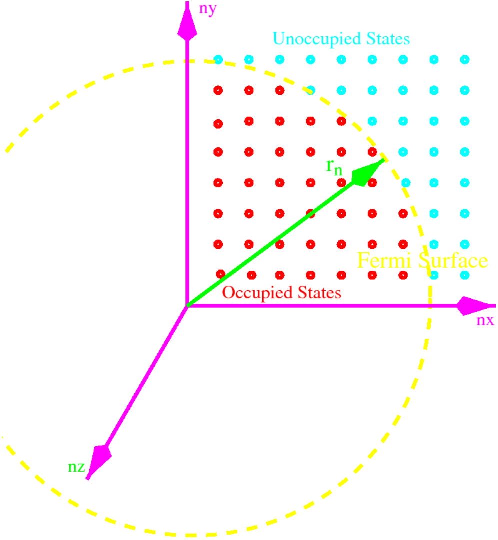
The trick

- Here is the trick. For each combination of (n_x, n_y, n_z) we have a single spacial state.
- The quantum numbers (n_x, n_y, n_z) change by one. A change by a unit in each direction define a cube of a unit volume. Thus we can state that there is one spacial state per unit volume in the space defined by (n_x, n_y, n_z) quantum numbers.
- Here is the crux: the volume in the (n_x, n_y, n_z) space is limited by the condition

$$n_x^2 + n_y^2 + n_z^2 < \frac{p_F^2 L^2}{\pi^2 \hbar^2} = R^2 \quad R = \frac{p_F L}{\pi \hbar} \tag{9}$$

- This condition defines a sphere with radius R in the (n_x, n_y, n_z) space.
- The volume of the sphere is the number of states up to the Fermi momentum since there is one state per unit volume.

The $(n_x, n_y, n_z$ space



The trick the right way

- This looked so good again. Something must be wrong.
- First, remember that $n_x > 0$ and $n_y > 0$ and $n_z > 0$, thus we have to take only $1/8$ of the sphere cut by positive x, y, and z axes.
- Next, nucleons have spins. Thus we have two states per unit volume as defined by two orientation of spins.
- With this corrections we can calculate the number of occupied states

$$n = 2\frac{1}{8}\frac{4}{3}\pi R^3 = \frac{1}{3}\pi\left(\frac{p_F L}{\pi\hbar}\right)^3 \tag{10}$$

- The Fermi momentum is

$$p_F = \hbar\sqrt[3]{3\pi^2}\frac{\sqrt[3]{n}}{L} = \hbar\sqrt[3]{3\pi^2}\sqrt[3]{\frac{n}{L^3}} = \hbar\sqrt[3]{3\pi^2}\sqrt[3]{\frac{n}{V}} \tag{11}$$

The Fermi momentum

- We have calculated the Fermi momentum p_F in terms of the density of states n/V

$$p_F = \hbar\sqrt[3]{3\pi^2}\frac{\sqrt[3]{n}}{L} = \hbar\sqrt[3]{3\pi^2}\sqrt[3]{\frac{n}{L^3}} = \hbar\sqrt[3]{3\pi^2}\sqrt[3]{\frac{n}{V}} \tag{12}$$

- Let us push it further assuming for the number of state the number of protons Z and for the volume

$$V = \frac{4}{3}\pi\left(r_0 A^{\frac{1}{3}}\right)^3 = \frac{4}{3}\pi r_0^3 A \tag{13}$$

- This implies

$$p_F = \hbar\sqrt[3]{3\pi^2}\sqrt[3]{\frac{n}{V}} = \frac{\hbar}{r_0}\sqrt[3]{\frac{3^2\pi}{4}}\sqrt[3]{\frac{Z}{A}} \tag{14}$$

Estimate for the Fermi momentum

- The Fermi momentum for protons is

$$p_F = \frac{\hbar}{r_0}\sqrt[3]{\frac{3^2\pi}{4}}\sqrt[3]{\frac{Z}{A}} \tag{15}$$

- The Fermi momentum for neutrons is

$$p_F = \frac{\hbar}{r_0}\sqrt[3]{\frac{3^2\pi}{4}}\sqrt[3]{\frac{A-Z}{A}} \tag{16}$$

- Assuming $Z/A \sim 1/2$ and $r_0 = 1.2$ fm the estimate for the Fermi momentum is

$$p_F = \frac{\hbar}{r_0}\sqrt[3]{\frac{3^2\pi}{4}}\sqrt[3]{\frac{Z}{A}} \approx 250 \;\; [\mathrm{MeV/c}] \tag{17}$$

- This corresponds to the speed

$$\beta_F = \frac{v_F}{c} \approx \frac{p_F c}{m_p c^2} \approx \frac{250}{940} = 0.27 \tag{18}$$

Estimate of the Fermi energy

- Fermi energy can be estimated from the Fermi momentum.
- Fermi energy depends on the density of states and is different for protons and for neutrons since in general there is a different number of particles in the proton and neutron well.
- The Fermi energy for protons is

$$E_F = \frac{p_F^2}{2m_p} = \frac{(\hbar c)^2}{2r_0^2 m_p c^2}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\left(\frac{Z}{A}\right)^{\frac{2}{3}} \quad (19)$$

- The Fermi energy for neutrons is

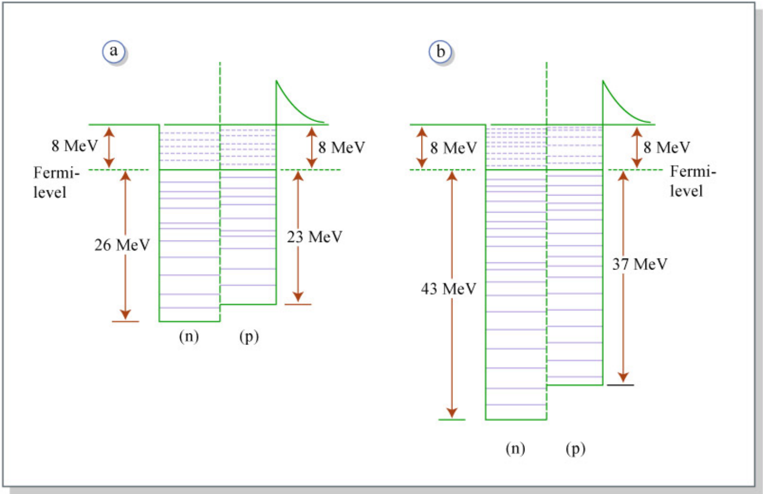
$$E_F = \frac{p_F^2}{2m_n} = \frac{(\hbar c)^2}{2r_0^2 m_n c^2}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\left(\frac{A-Z}{A}\right)^{\frac{2}{3}} \quad (20)$$

- For the case of even number of protons and neutrons $Z/A \approx (A-Z)/A \approx 1/2$ and $r_0 = 1.2$ fm the Fermi energy is $E_F \sim 33$ MeV.

The depth of the nuclear potential well

- From the Fermi energy we know (approximately) the energy difference between the bottom of the potential well and the energy of the highest occupied state.
- We also know from the binding energies that the last nucleon is bound by approximately 8 MeV.
- From that we can conclude that the depth of the nuclear potential well is approximately 33+8=41 MeV.
- We get the same depth for protons and neutron since we assumed the equal number of both.
- If we take into account the fact that the difference in the number of protons and neutrons we will get different depths of the potential wells for each species.

The depth of the nuclear potential well

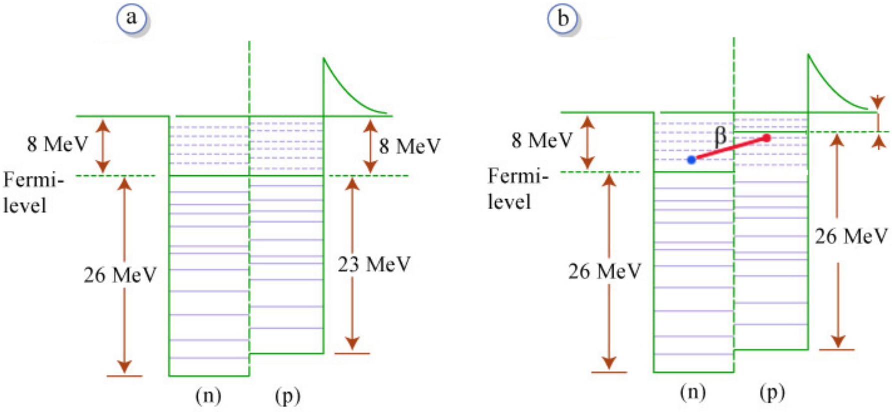


The depth difference for protons and neutrons

- The potential depth difference comes out from the dependence of the Fermi energy on the number of protons or neutrons.
- The difference in depth is a consequence of the Coulomb repulsion between the protons.
- Note the difference between the depth of the potential well and the Fermi level (energy of the highest occupied state).
- In stable nuclei the Fermi level is at the same energy for protons in neutrons. In such a case there is no energy gain from transforming one type of nucleons into another by β decay.
- In unstable nuclei the Fermi level is different for protons and neutrons, this opens a path to transform nucleons from one well to the other through β decay.
- The β decay proceeds until Fermi levels in both wells are equal.

The Fermi level

Fermi level for protons in neutrons
in stable nuclei (a) and in unstable nuclei (b).



Average momentum of a nucleon

- The average momentum of a nucleon depends on the density of momentum states which is the number of momentum states dn for a particle with momentum p and $p + dp$.
- We can calculate this number using a similar trick to what we used to calculate the number of states up to the Fermi momentum.
- Let us consider the space defined by the quantum numbers (n_x, n_y, n_z). We argued there are two momentum states per unit volume in this space.
- Particles with momentum between p and $p + dp$ define a spherical shell with radius $R = pL/\pi\hbar$ and thickness $dR = dpL/\pi\hbar$ in this space with volume

$$\mathcal{V} = 4\pi R^2 dR = 4\pi \left(\frac{L}{\pi\hbar}\right)^3 p^2 dp = \frac{4V}{\pi^2\hbar^3} p^2 dp \qquad (21)$$

Average momentum of a nucleon

- To get the number of states we need to remember that the condition for $n_x > 0$, $n_y > 0$ and $n_z > 0$ requires that we take $1/8$ of the volume $\mathcal{V}$, and that we also need to multiply $\mathcal{V}$ by 2 to account for two spins state of the nucleon.
- The density of states is

$$\begin{aligned} dn &= 2\frac{1}{8}\mathcal{V} = \frac{1}{4}\frac{4V}{\pi^2\hbar^3}p^2dp = \frac{V}{\pi^2\hbar^3}p^2dp \\ \frac{dn}{dp} &= \frac{V}{\pi^2\hbar^3}p^2 \end{aligned} \tag{22}$$

- The average momentum of a nucleon is

$$p_{av} = \frac{\int_0^{p_F} p\frac{dn}{dp}p^2dp}{\int_0^{p_F} \frac{dn}{dp}p^2dp} = \frac{\int_0^{p_F} p^3dp}{\int_0^{p_F} p^2dp} = \frac{3}{4}p_F \approx 188 \ \text{MeV} \tag{23}$$

The average energy of a nucleon

- Watch out! The average energy is not the energy for the average momentum!
- To calculate the average energy we need to know the density of energy state, which is the number of states for a particle with energy between E and dE.
- We can get this number from the density of the momentum states

$$
\begin{aligned}
dn &= \frac{V}{\pi^2\hbar^3}p^2 dp \\
E &= \frac{p^2}{2m} \implies p^2 = 2mE \implies pdp = mdE \\
dp &= \frac{mdE}{\sqrt{2mE}} = \sqrt{\frac{m}{2}}\frac{dE}{\sqrt{E}} \\
dn &= \sqrt{2}\frac{V}{\pi^2\hbar^3}m^{\frac{3}{2}}\sqrt{E}dE \qquad (24)
\end{aligned}
$$

The average energy of a nucleon

- The average energy of a nucleon is

$$E_{av} = \frac{\int_0^{E_F} E \frac{dn}{dE} dE}{\int_0^{E_F} \frac{dn}{dE} dE} = \frac{\int_0^{E_F} E^{\frac{3}{2}} dE}{\int_0^{E_F} E^{\frac{1}{2}} dE} = \frac{3}{5} E_F \approx 20 \ \text{MeV} \qquad (25)$$

- The energy which corresponds to the average momentum is

$$E' = \frac{p_{av}^2}{2m} = \frac{9}{16} \frac{p_F^2}{2m} = \frac{9}{16} E_F \approx 18.6 \ \text{MeV} \qquad (26)$$

Total energy in a nucleus

- Knowing the average energy of a nucleon we can calculate the total energy of a nucleus

$$E_{tot} = NE^{\nu}_{av} + ZE^{\pi}_{av} = N\frac{3}{5}E^{\nu}_{F} + Z\frac{3}{5}E^{\pi}_{F} \tag{27}$$

- Assuming for simplicity equal mass for a proton and a neutron

$$E_{tot} = \frac{3}{10}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \frac{\hbar^2 c^2}{mc^2 r_0^2} \frac{N^{\frac{5}{3}} + Z^{\frac{5}{3}}}{A^{\frac{2}{3}}} \tag{28}$$

- Let us investigate the term which depends on N, Z and A further as it leads to some interesting consequences.

Total energy in a nucleus

- The term dependent on N, Z and A expands to

$$\frac{N^{\frac{5}{3}} + Z^{\frac{5}{3}}}{A^{\frac{2}{3}}} = A\left(1 + \frac{5}{9}\left(\frac{N-Z}{A}\right)^2\right) \quad (29)$$

- Therefore the total energy has two terms

$$E_{tot} = \frac{3}{10}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \frac{\hbar^2 c^2}{mc^2 r_0^2} A + \frac{1}{6}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \frac{\hbar^2 c^2}{mc^2 r_0^2}\left(\frac{N-Z}{A}\right)^2 \quad (30)$$

- The first term corresponds to the volume term in the liquid drop model while the last one corresponds to the symmetry energy term in the liquid drop model.
- Therefore, based on the Fermi model we can conclude that the symmetry energy term in the liquid drop model is a quantum mechanical effect related to the way fermion occupy allowed states in the proton/neutron potential well.