

Collective model

Introduction to Nuclear Science

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Outline

- 1 Single particle vs. collective behaviour

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- 2 Deformation of a sphere

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- 3 Surface vibrations

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- 4 Static deformation

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- 4 Static deformation
- 5 Euler angles

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- 4 Static deformation
- 5 Euler angles
- 6 Intrinsic reference frame

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- 7 Nuclear rotor

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- 4 Static deformation
- 5 Euler angles
- 6 Intrinsic reference frame
- 7 Nuclear rotor
- 8 Quadrupole deformation parameters

Nuclear single-particle behaviour

- Spherical and deformed shell models view nucleus as a collection of fermions occupying single-particle states in a potential well.
- These models can successfully predict properties of nuclear states with configuration dominated by a single nucleon, or, to some extent, by relatively small number of nucleons.
- We discussed example application of these models in predicting spins, parities, magnetic, and quadrupole moments of states with a single nucleon, or a pair of nucleons outside an even-even core.
- Nuclear properties which are determined by a single nucleon are often referred to as the single-particle properties.
- In numerous cases nuclear behaviour can be described in terms of the single particle properties, however, in equally numerous examples the single particle description is far from being adequate.

Nuclear collective behaviour

- Experimental data suggest that in many nuclei ground states or low-energy excitations involve a coordinated, large-amplitude motion of many nucleons.
- Nuclear properties which are determined by such a coordinated, large-amplitude motion of many nucleons are often referred to as collective properties.
- Surprisingly, collective properties are often quite simple to describe in terms of deformation of nuclear surface.
- Examples are provided by nuclear vibrations around spherical shape or nuclear rotation of a deformed shape.
- Note, however, that explaining nuclear collective phenomena in terms of the single-particle motion can be very complicated, if at all possible.

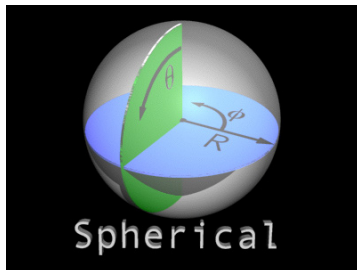
Comments on collective and single-particle behaviour

- Nuclear surface or nuclear potential are good examples of a collective behaviour in itself.
- Indeed, we discussed the nuclear potential as resulting from averaged two-body interactions of all nucleons in nuclei.
- Nuclear surface, in turn, results from a superposition of density distribution of all nucleons in a nucleus.
- Both are in principle very complicated if addressed in terms of the superposition of single-particle properties but have simple and intuitive interpretation in terms of the collective properties.
- Nuclei provide numerous examples of emergence of simple patterns in complex systems which is an active part of research in nuclear science.

Spherical shell

- Let us first consider a spherical shell.
- The shell is defined by a constant radius.
- A spherical shell is easily defined in the spherical coordinate system by

$$R(\theta, \phi) = R_0 = \text{const.} \quad (1)$$



Deformed shell

- Let us now deformed the shell by changing the radius slightly with respect to R_0 .
- This, in principle, can be a complicated task since now the radius becomes a function of the polar and azimuthal angles θ and ϕ .
- Any deformation of the spherical shell can be represented as a series of spherical harmonics which are functions of the polar and azimuthal angles θ and ϕ .
- The spherical harmonics are the same functions which represent the orbital motion in the shell model. Recall that they have a rank l and m $Y_{l,m}(\theta, \phi)$ and spherical harmonics of different rank are different functions of the polar and azimuthal angle.
- In the shell model the rank l and m defined the orbital angular momentum and the magnetic quantum number. Here, the rank defines different deformation of the shell.

Deformed shell

- The deformed shell as a function of the polar and azimuthal angles is defined by

$$\begin{aligned}
 R(\theta, \phi) &= R_0 [1 + \\
 &\quad \alpha_{1,-1} Y_{1,-1}(\theta, \phi) + \alpha_{1,0} Y_{1,0}(\theta, \phi) + \alpha_{1,1} Y_{1,1}(\theta, \phi) + \\
 &\quad \alpha_{2,-2} Y_{2,-2}(\theta, \phi) + \alpha_{2,-1} Y_{2,-1}(\theta, \phi) + \\
 &\quad \alpha_{2,0} Y_{2,0}(\theta, \phi) + \alpha_{2,1} Y_{2,1}(\theta, \phi) + \alpha_{2,2} Y_{2,2}(\theta, \phi) + \dots] \\
 &= R_0 \left[1 + \sum_{l=1}^{\infty} \sum_{m=-l}^l \alpha_{l,m} Y_{l,m}(\theta, \phi) \right] \quad (2)
 \end{aligned}$$

- In this expansion coefficient $\alpha_{l,m}$ is the amplitude of the deformation defined by the spherical harmonics of the rank l, m . For applet showing deformations of a particular rank [this link](#)

Nuclear deformation

- The expansion of the deformed shape into spherical harmonics is used in many field of science and can be done for gravity, electromagnetic fields, liquid drop, as well as for deformed nuclear shapes.
- In the nuclear case the important factor is the incompressibility of the nuclear matter which implies volume conservation.
- Moreover the nuclear deformation should not change the position of the centre of mass of a nucleus.
- As we argued in an example given in lecture 3 for small deformations the rank $l = 1$ (dipole) deformation shifts the centre of mass without changing the shape.
- Thus, the amplitudes which define independent deformations are these for rank $l = 2$ (quadrupole) and higher.
- The coefficients for the dipole deformation are adjusted to ensure the constant volume and position of the centre of mass.

Time dependent amplitudes

- Let us consider the case of small deformations with time dependent amplitudes $\alpha_{l,m}(t)$.
- The conservation of nuclear volume imposes conditions on these amplitudes. There is a restoring force preventing these amplitudes to grow large and driving the shape of a nucleus towards sphericity.
- This is completely analogous to the surface vibration of the droplet of a liquid around the spherical shape.
- In the nuclear case the restoring force can be estimated based on the liquid drop model by calculating change in the energy associated with surface deformation at a constant volume.
- In the lowest order approximation time-dependent amplitudes describe harmonic oscillations of the nuclear surface around the spherical shape.

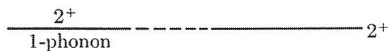
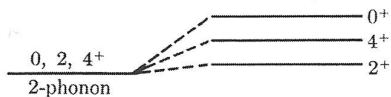
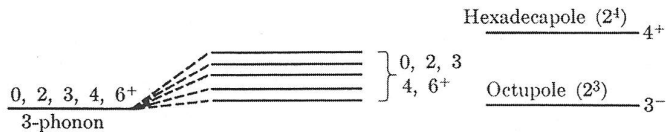
Harmonic vibrations of various ranks

- This applet shows harmonic surface vibrations of various ranks, starting with
 - 1 $l = 0$ (monopole breathing mode) violating volume conservation
 - 2 $l = 1$ (dipole) shifting centre of mass
 - 3 $l = 2$ (quadrupole)
 - 4 $l = 3$ (octupole)
 - 5 $l = 4$ (hexadecupole)
- Vibrational mode of surface at a given rank is called a phonon of a given rank (for example quadrupole phonon or octupole phonon).
- Phonon of a given rank carries angular momentum l and parity $(-1)^l$ defined by its rank.
- Low energy excitations in vibrational nuclei are defined by the number of excited phonons and coupling of their angular momenta.

The phonon model

- The phonon model explains low-energy collective excitations of nuclei as resulting from surface vibrations. In this model there is an energy associated with an excitation of a phonon.
- The ground state at zero energy is the zero-phonon state.
- The first excited state is the one-phonon state. The excitation energy of this state is equal to the excitation energy of the phonon. The spin and parity of that state is the rank and parity of the phonon.
- For example for a quadrupole vibrator the first excited state has spin 2 and parity $(-1)^2 = +1$. For an octupole vibrator the first excited state has spin 3 and parity $(-1)^3 = -1$.
- The next group of excited states are the two-phonon states. Excitation energy is twice of the first excited state. Spins/parity of these states are defined by the coupling of two phonon angular momenta, in general from 0^+ to $2I^+$.

The phonon model



Pure
vibrational

(a)

Vibrational
states in
real nuclei

(b)

Higher-order
vibrations

(c)

The phonon model

- In general, the energy of the phonon grows with its rank. Therefore the energy of a quadrupole phonon is smaller than the energy of an octupole phonon, etc.
- In the simplest version the phonon model predicts that the energy for a multi-phonon state is the number of phonons times the energy of a single phonon.
- This is true for non-interacting phonons. If phonons interact, multiphonon states of different spin are shifted in various degrees by the interactions.
- These shifts are evidence of non-harmonicity in the phonon model.
- Data on vibrational nuclei indicate a large degree of anharmonicity.

Anharmonicity in ^{112}Cd

0⁺ 1432

4⁺ 1416

2⁺ 1312

2⁺ 617

0⁺
0

The phonon model

- The phonon model predicts existence of electromagnetic transitions between the ground state and excited states.
- On the absorption of a photon by a vibrational nucleus the photon energy is converted into the phonon energy (phonon excitation). The energy of the photon has to be equal to the energy of the phonon plus recoil energy of the nucleus.
- A phonon state can also de-excite with an emission of a photon. The number of phonons in the deexcitation decreases by one, the energy of the photon is the energy of the phonon less the recoil energy of the nucleus.
- Note that electromagnetic transitions can change the number of phonons by one only. Transitions changing the number of phonons by more or less than one are forbidden.

Static deformation

- Let us come back to the expansion of the radius in terms of spherical harmonics

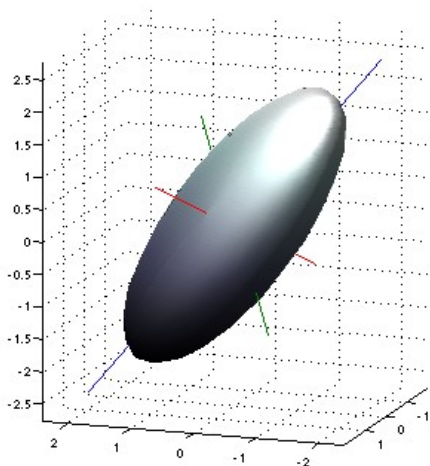
$$R = R_0 \left[1 + \sum_{l=1}^{\infty} \sum_{m=-l}^l \alpha_{l,m} Y_{l,m}(\theta, \phi) \right] \quad (3)$$

- If we fix the axis of the coordinate frame and if the amplitudes α are constant in time we have a case of static deformation.
- Static deformations of a globe with fixed axes were shown by [the applet](#) we discussed earlier.
- We also discussed in Lecture 3 that an object with static deformation defines a coordinate system in space.
- In particular we have discussed the system with ellipsoidal (quadrupole) deformation.

Euler angles

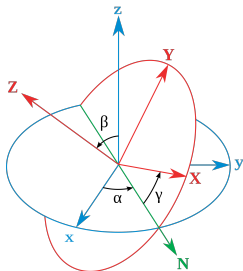
- For a deformed object we can distinguish two reference frames
 - ① Ours, which we will referred to as the lab reference frame.
 - ② Objects, which we will referred to as the intrinsic or the body-fixed reference frame.
- For example, for a triaxial potential well the intrinsic reference frame is defined by the three axes of the well. The same is true for triaxial ellipsoidal deformation.
- It is important to recognize that for nuclei the intrinsic reference frame can have any orientation with respect to the lab reference frame as we can hardly control orientation of nuclei (although it is possible in some cases).

Intrinsic (body-fixed) and lab-fixed referenced frames



The Euler angles

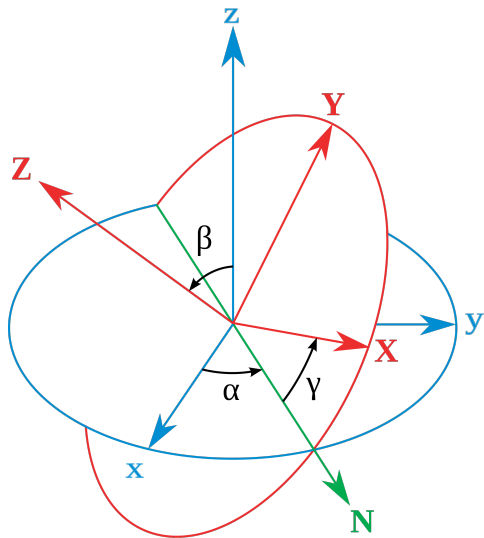
- It is important to recognize that for nuclei the intrinsic reference frame can have any orientation with respect to the lab reference frame as we can hardly control orientation of nuclei (although it is possible in some cases).
- One way to specify the mutual orientation of two reference frames of the common origin is to use the Euler angles.



The Euler angles

- Here are the steps to define Euler angles between the (x, y, z) axes of the lab frame (red) and the (x', y', z') axes of the intrinsic frame (blue).
 - 1 Define the line of nodes (green) as $\vec{N} = \vec{z} \times \vec{z}'$, this line is perpendicular to the plane defined by the z and the z' axes of both coordinate frames.
 - 2 Angle α is between the x' axis and the line of nodes.
 - 3 Angle β is the angle between the z' and z axis.
 - 4 Angle γ is the angle between the line of nodes and the x axis.
- Rotation by angle α about the z' axis, β about the x' axis and γ about the z' axis (which at the time of this rotation overlaps with the z axis) brings the intrinsic frame onto the lab frame.

The Euler angles



Quadrupole rotor vs. nuclear vibrator

- For simplicity, let us concentrate on the first independent deformation which is of rank 2 (quadrupole).
- Remember that the dipole is not independent as it moves the centre of mass and changes volume.
- In the nuclear rotor model the Euler angles become time dependent coordinates while the deformation parameters in the intrinsic system are static. See [this applet](#).
- This should be contrasted with the vibrator model in which the deformation parameters are time-dependent but the intrinsic frame and Euler angles can not be defined. See [this applet](#).

Quadrupole deformation parameters

- Let us try to estimate how many parameters are needed to describe static quadrupole deformation.
- Let us start with the number of amplitudes: there are five coefficients for the deformation of rank 2: $\alpha_{2,-2}$, $\alpha_{2,-1}$, $\alpha_{2,0}$, $\alpha_{2,1}$, $\alpha_{2,2}$
- These coefficients are in principle complex, implying 10 coefficients.
- But the surface is real. This fact imposes a conditions for the amplitudes reducing the number of independent coefficients to five.
- Furthermore, there is a relation between these five amplitudes and three Euler angles defining the orientation of the intrinsic coordinate frame.
- For quadrupole only two amplitudes are truly independent. Thus two parameters fully define quadrupole deformation in the intrinsic reference frame.

Bohr-Wheeler parametrization

- The two parameters of static quadrupole deformation in the intrinsic system are often chosen as the Bohr-Wheeler parameters β and γ .
- The deformed surface in terms of these parameters is defined as

$$R(\theta, \phi) = R_0(1 + \beta \cos \gamma Y_{2,0}(\theta, \phi) + \frac{1}{\sqrt{2}}\beta \sin \gamma (Y_{2,2}(\theta, \phi) + Y_{2,-2}(\theta, \phi))) \quad (4)$$

- The axes lengths of the ellipsoid in the intrinsic reference frame are

$$R_{x'} = R_0(1 + \sqrt{\frac{5}{4\pi}}\beta \cos(\gamma - 120^\circ))$$

$$R_{y'} = R_0(1 + \sqrt{\frac{5}{4\pi}}\beta \cos(\gamma + 120^\circ))$$

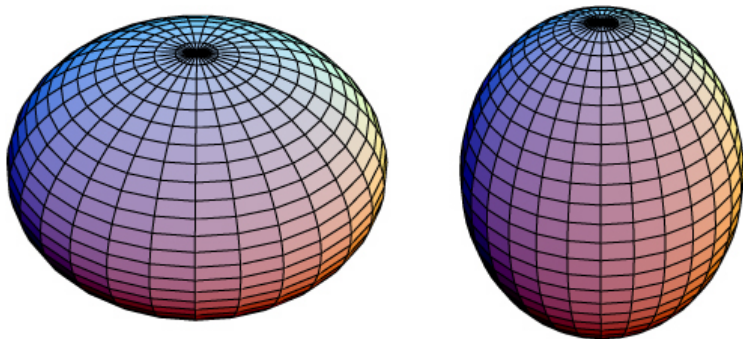
$$R_{z'} = R_0(1 + \sqrt{\frac{5}{4\pi}}\beta \cos(\gamma))$$

Axial quadrupole deformation

- In the Bohr-Wheeler parametrization the parameter β defines the elongation along the z' axis in the intrinsic reference frame.
- The parameter γ defines the length difference between the x' and the y' axes of the quadrupole-deformed shape in the intrinsic frame.
- Thus the parameter γ defines the triaxiality of the shape.
- Let us consider the special case of $\gamma = 0^\circ$). For $\gamma = 0^\circ$ $R_{x'} = R_{y'}$ and the shape is axially symmetric with the z' axis being the axis of symmetry.
- For the axial case of $\gamma = 0^\circ$ positive values of β correspond to the prolate deformation while negative values of β correspond to the oblate deformation.

Axial quadrupole deformation

Quadrupole deformation (Left) oblate, $\beta < 0$ (Right) prolate, $\beta > 0$



Axial rotor

- So far everything looks dandy. Something must be wrong!
- Some of you may, and all of you should feel uncomfortable by now due to the fact that there is no clear way to define the intrinsic system for an axial rotor.
- The z' axis is fine, this is the symmetry axis of the deformation.
- But what about the x' and y' axes? For an axially symmetric deformation there is no clear way to define the direction of the x' and the y' axes.
- As a matter of fact there is an infinite number of ways to do it, anyone of them equivalent to any other.
- This has a profound consequences for the rotor.

Quantum quadrupole axial rotor

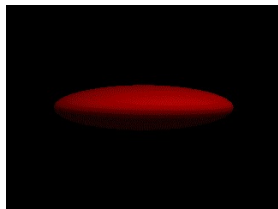
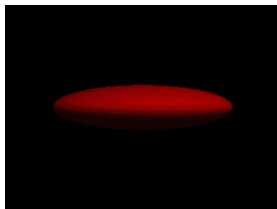
- The first consequence for an axial rotor in quantum mechanical description is the observation that rotation around the symmetry axis does not result in a new state.
- Truly, the rotation around the symmetry axis changes the phase factor of the wave function only.
- Since the energy of a state does not depend on the phase factor (depends on the wave function squared) all the states which are related by rotation around the symmetry axis have the same energy.
- This means that there are no quantum mechanical excitations related to rotation about the symmetry axis.
- The excitation must then involve rotation about the axis perpendicular to the symmetry axis.

Quantum quadrupole axial rotor

- The property of the quantum axial rotor defined in the previous slide is sometimes expressed by stating that “quantum mechanics” forbids rotation around a symmetry axis.
- This is clearly wrong, the rotation is allowed it generates the phase factor but does not generate excitations.
- In fact, rotations shown in [this applet](#), [this applet](#), and [this applet](#) in quantum mechanics correspond to the same single state and the same energy, independent of rotational frequency.
- In contrast, the rotation show in [this applet](#) corresponds to different energy states if the rotational frequency is different.

Quantum quadrupole axial rotor

- From now we rotate perpendicular to the symmetry axis.
- The next property of the quantum quadrupole axial rotor to be recognized is a symmetry with respect to the rotation by 180° about the axis perpendicular to the symmetry axis.
- In quantum mechanics there is no distinction between a quantum quadrupole axial rotor and a quantum quadrupole axial rotor rotated by 180° around the symmetry axis.



$$E_I = \frac{\hbar^2}{2I} I(I+1) \quad (6)$$

Quantum quadrupole axial rotor

- Rotational excitation energies of a quantum quadrupole axial rotor are
- I is the angular momentum (spin) of the state, J is the moment of inertia.
- As a consequence of the 180° symmetry with respect to rotation about the axis perpendicular to the symmetry axis only even values of I are allowed for excited levels.
- Consequently the energy levels are

Spin I	0	2	4	6	8
Energy E	0	$6 \frac{\hbar^2}{2J}$	$20 \frac{\hbar^2}{2J}$	$42 \frac{\hbar^2}{2J}$	$72 \frac{\hbar^2}{2J}$

Analogy with di-atomic molecules

- Rotational spectra for nuclear quantum quadrupole axial rotor are analogues to the rotational spectra of di-atomic molecules with both atoms being of the same kind, like for example H_2 .
- In the di-atomic molecules with both atoms of the same kind there is the same restriction for rotation only about the axis perpendicular to the symmetry axis, and the symmetry of rotation by 180° around the axis perpendicular to the symmetry axis.
- Consequently, the di-atomic molecules of this type have the excitation spectra as defined on the previous slide with even-spin states only.

Analogy with di-atomic molecules

- Rotational excitation spectra are different for di-atomic molecules with two atoms of a different kind, like for example *HCl*.
- The energies are still given by

$$E_l = \frac{\hbar^2}{2J} l(l+1) \quad (7)$$

- For *HCl*, however, there is no symmetry between the rotation by 180° about the axis perpendicular to the symmetry axis, since this rotation exchanges *H* with *Cl*
- Consequently, odd-spin state are allowed and present in excitation spectrum of *HCl*.
- Nuclear analog of *HCl* is a rotor with axial octupole deformation.