Rutherford scattering

Introduction to Nuclear Science

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Outline

1. Scattering via the Coulomb force

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Charged-particle induced reactions

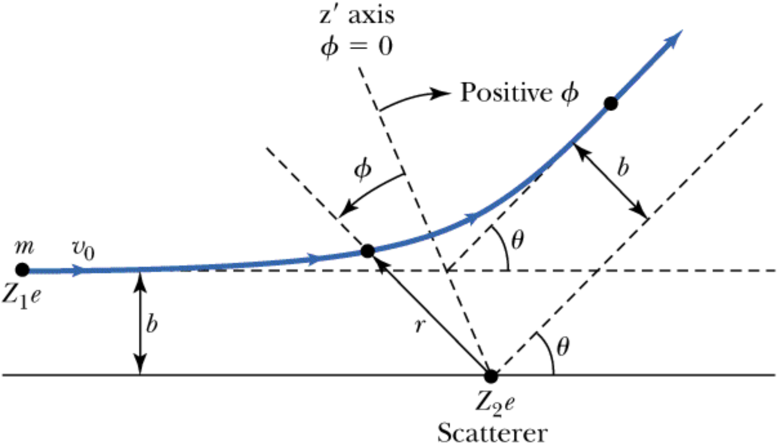
- Nuclei contain protons and are charged with the charge of $+Ze$.
- There is a long-range repulsive force between reacting nuclei:

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{Z_1 Z_2 e^2}{r^2} \frac{\vec{r}}{r} \tag{1}$$

- Since the force is repulsive the trajectories are hyperbolic.
- If projectile energy is not sufficient to bring two nuclei to a distance smaller than the range of nuclear interactions, the only result of the collision is either elastic (Rutherford) or inelastic (Coulomb) scattering.
- For Rutherford scattering both projectile and target emerge from the collision in their respective ground states. In Coulomb scattering either the target or the projectile emerge in an excited state.

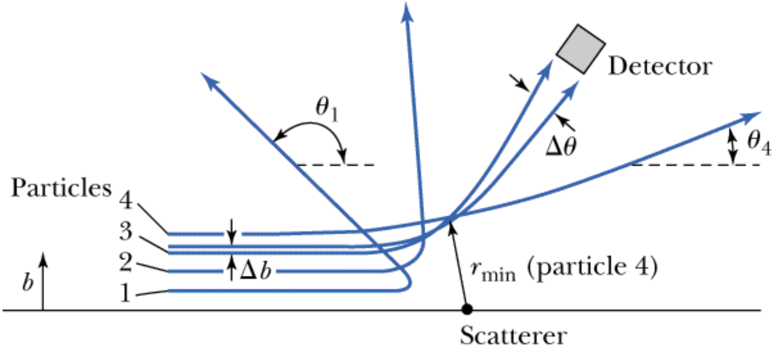
Trajectories in Rutherford scattering

- Trajectories are hyperbolic.



Trajectories in Rutherford scattering

- Trajectories depend on the impact parameter.



The Geiger-Marsden experiment

- Based on the knowledge of
  - the Coulomb force,
  - the Newton's laws,
  - the conservation of linear momentum,
  - the conservation of angular momentum

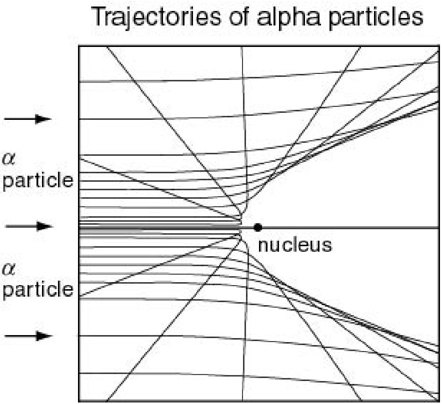
  Rutherford in 1911 derived the formula describing the number of α-particles scattered from a thin gold foil at a given laboratory angle.
- This formula fitted very well the experimental data taken by Rutherford's students Geiger and Marsden.
- The derivation assumed that the whole mass of the gold atom and the α particle was concentrated in a very small, point-like volume.

The ironic ways of Nature

- The agreement of the Rutherford formula with experimental data of Geiger and Marsden implicated concentration of almost entire mass of an atom in the nucleus of a femtometer size.
- Thus Rutherford discovered 100 years ago the atomic nucleus.
- Another Rutherford's student, Niels Bohr proposed in 1913 the model of Hydrogen atom based on the idea of the massive nucleus in the centre but augmented in quantization conditions to explain the observed Hydrogen line spectra.
- The Bohr atom model jump started modern Quantum Mechanics
- But Rutherford derived his formula based on Classical Mechanics, which truly is not applicable to the scattering of elementary particles.
- Fortunately, the Classical and Quantum Mechanics lead to the same result for Rutherford scattering. Otherwise, the history of Science could have been very different.

Trajectories in Rutherford scattering

- Trajectories depend on the impact parameter.



Trajectories in Rutherford scattering

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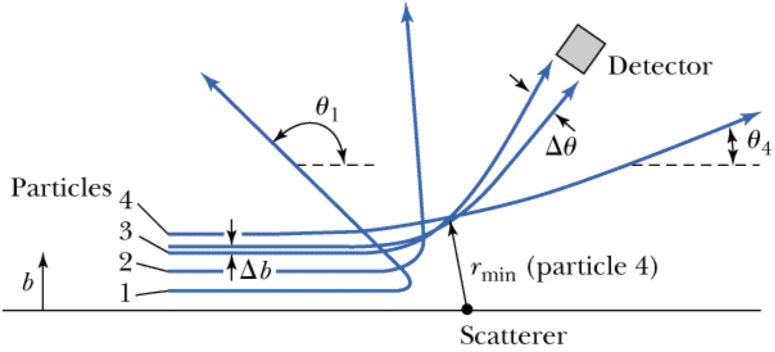
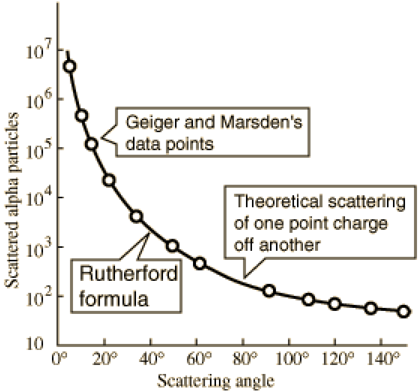


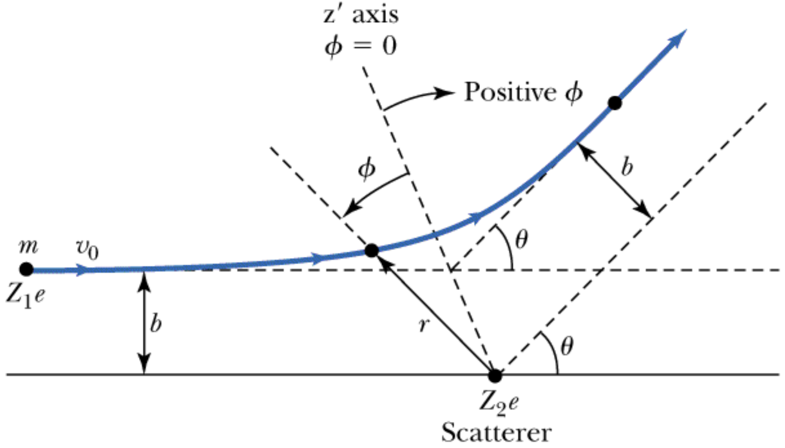
The Geiger-Marsden experiment

- Number of particles scattered at a given angle in Rutherford scattering is calculable and well understood, since it is defined by the well understood electromagnetic force.



The impact parameter/scattering angle relationship

- The key concept in Rutherford scattering is the relationship between the impact parameter b and the scattering angle θ.



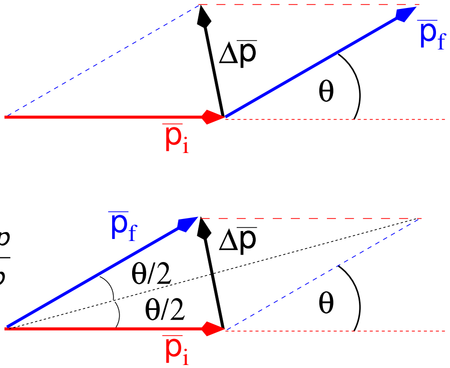
The impact parameter/scattering angle relationship

- Let us examine the change of momentum in the scattering.
- The magnitude of the momentum stays the same, since this is the elastic scattering.
- The direction of the momentum is changed since this is a scattering.
- The change of the direction results from Coulomb repulsion.
- We assume after Rutherford that the target is very heavy and therefore stationary during the scattering process.
- More sophisticated calculations are done by making centre of mass transformation.

Momentum change in Rutherford scattering

- The momentum change is

$$| \vec{p_i} |=| \vec{p_f} |= p$$
$$| \Delta\vec{p} |= \Delta p$$
$$\sin(\theta/2) = \frac{\frac{1}{2}\Delta p}{p} = \frac{\Delta p}{2p}$$
$$\Delta p = 2p\sin(\theta/2)$$



The momentum change

- From the Newton's second law

$$\vec{F} = \frac{d\vec{p}}{dt} \implies \Delta\vec{p} = \int \vec{F} dt \tag{2}$$

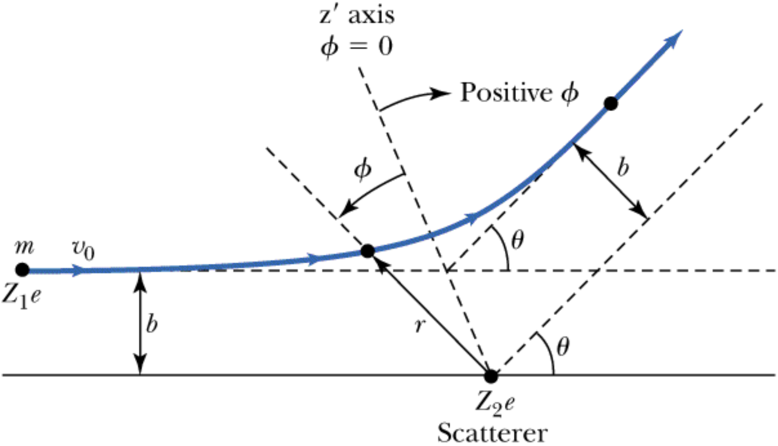
- The force is the Coulomb force

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{Z_1 Z_2 e^2}{r^2} \frac{\vec{r}}{r} \tag{3}$$

- Before we start integrating let us note that the trajectories are symmetric with respect to the line defined by the distance of the closest approach.

Trajectories in Rutherford scattering

- Trajectories are symmetric with respect to angle ϕ.



The momentum change

- The symmetry with respect to the line at $\phi = 0$ implies

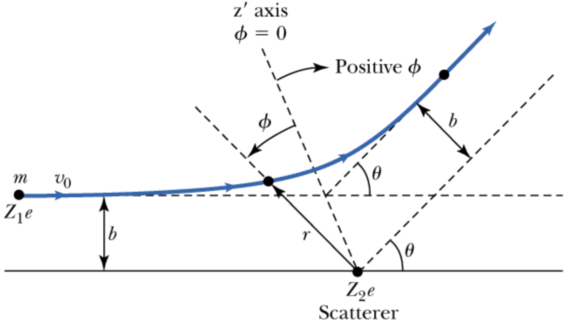
$$\Delta\vec{p} = \int \vec{F} dt \Longrightarrow \Delta p = \int F \cos\phi dt$$
$$\Delta p = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0} \int \frac{1}{r^2} \cos\phi dt \quad (4)$$

- This integral can be carried over with a help of conservation of angular momentum.

The angular momentum

- The angular momentum is

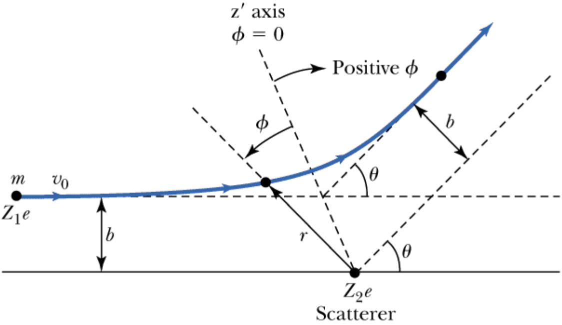
$$\begin{aligned} \vec{L} &= \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} = m\vec{r} \times \left(\frac{d\vec{r}}{dt} + r\frac{d\vec{\phi}}{dt}\right) = mr\vec{r} \times \frac{d\vec{\phi}}{dt} \\ L &= \mid \vec{L} \mid = mr^2 \frac{d\phi}{dt} \end{aligned} \tag{5}$$



The angular momentum

- But also from the initial condition

$$L = mv_0 b \tag{6}$$



Conservation of angular momentum

- Since the angular momentum is conserved

$$\begin{aligned} mr^2 \frac{d\phi}{dt} &= mv_0 b \\ \frac{dt}{r^2} &= \frac{d\phi}{v_0 b} \end{aligned} \tag{7}$$

- Thus the change of momentum is

$$\begin{aligned} \Delta p &= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0} \int \frac{dt}{r^2} \cos\phi = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0} \int \frac{d\phi}{v_0 b} \cos\phi = \\ &= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0} \frac{1}{v_0 b} \int_{\phi_<}^{\phi_>} \cos\phi d\phi \end{aligned} \tag{8}$$

Integration limits

- The limits for integration are defined by

$$\begin{aligned} \phi_< + \phi_> + \theta &= \pi \\ \phi_< &= -\phi_> \end{aligned} \tag{9}$$

- The solution to these equations is

$$\begin{aligned} \phi_< &= -\frac{1}{2}(\pi - \theta) \\ \phi_> &= \frac{1}{2}(\pi - \theta) \end{aligned} \tag{10}$$

- The integral is

$$\begin{aligned} \Delta p &= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{v_0 b}\int_{\phi_<}^{\phi_>} \cos\phi d\phi = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{v_0 b}(\sin\phi_> - \sin\phi_<) \\ &= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{v_0 b} 2\sin(\frac{1}{2}(\pi - \theta)) = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{2}{v_0 b}\cos(\theta/2) \end{aligned} \tag{11}$$

The impact parameter/scattering angle relationship

- Combining Eq.2 with Eq.11 yields

$$
\begin{aligned}
\Delta p &= 2p\sin(\theta/2) = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{2}{v_0 b}\cos(\theta/2) \\
b &= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{p v_0}\frac{1}{\tan(\theta/2)} \\
b &= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{2K}\frac{1}{\tan(\theta/2)} \qquad (12)
\end{aligned}
$$

with K being the initial kinetic energy for the projectile.

- Equation 12 defines the relationship between the impact parameter and the scattering angle.

The distance of the closest approach

- Assuming a head on collision the distance of the closest approach d_0 can be calculated from the conservation of energy

$$K = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{d_0}$$
$$d_0 = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{K} \tag{13}$$

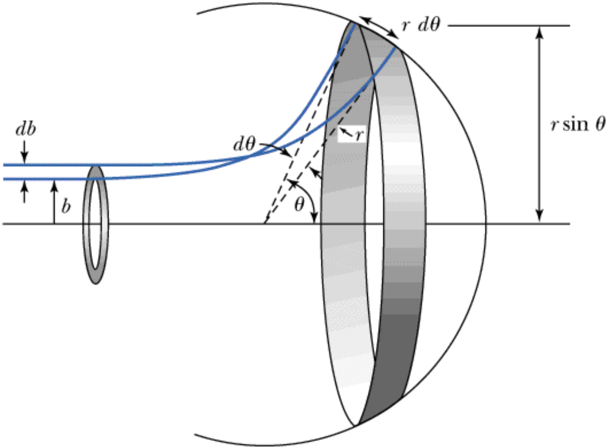
- The relation between the impact parameter and the scattering angle simplifies with the use of the distance of the closest approach parameter

$$b = \frac{d_0}{2}\frac{1}{\tan(\theta/2)} \tag{14}$$

- The distance of the closest approach is sometimes called the Sommerfeld parameter.

The cross section in Rutherford scattering

- Particles from the ring defined by the impact parameters b and $b + db$ scatter between angles θ and $\theta - d\theta$



The cross section

- Since particles from the ring defined by the impact parameters b and $b + db$ scatter between angles θ and $\theta + d\theta$ the cross section for scattering into the angle θ (called the differential cross section) is

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b db}{2\pi sin\theta d\theta} = \frac{b}{\sin\theta}\frac{db}{d\theta} \tag{15}$$

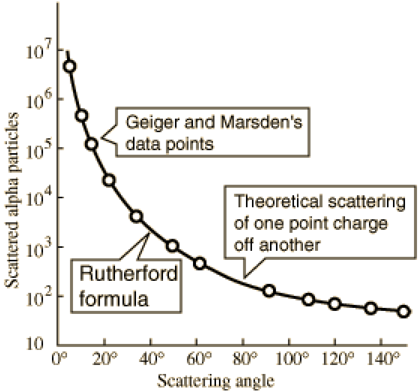
- The relationship between b and θ for the Rutherford scattering yields

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{d_0}{4}\right)^2 \frac{1}{\sin^4(\theta/2)} \\ &= \left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{4K}\right)^2 \frac{1}{\sin^4(\theta/2)} \end{aligned} \tag{16}$$

- This is the Rutherford result explaining the Geiger-Marsden experiment

The Geiger-Marsden experiment

- Number of particles scattered at a given angle in Rutherford scattering is calculable and well understood, since it is defined by the well understood electromagnetic force.



The distance of closest approach revisited

- The distance of closest approach d as a function of the impact parameter can be calculated from the conservation of energy, now withouth the assumption of the head-on collision

$$\frac{1}{2}mv_0^2 = \frac{1}{2}v^2 + \frac{Z_1Z_2e^2}{4\pi\varepsilon_0}\frac{1}{d} \tag{17}$$

- With a bit of algebra the above equation yields

$$\left(\frac{v}{v_0}\right)^2 = 1 - \frac{d_0}{d} \tag{18}$$

- Moreover for the distance of the closest approach the conservation of angular momentum yields

$$mv_0 b = mvd \tag{19}$$

The distance of closest approach revisited

- From the conservation of angular momentum

$$b^2 = d^2 \left(\frac{v}{v_0} \right)^2 \tag{20}$$

- Combining with the conservation of energy

$$b^2 = d^2(1 - \frac{d_0}{d}) = d(d - d_0) \tag{21}$$

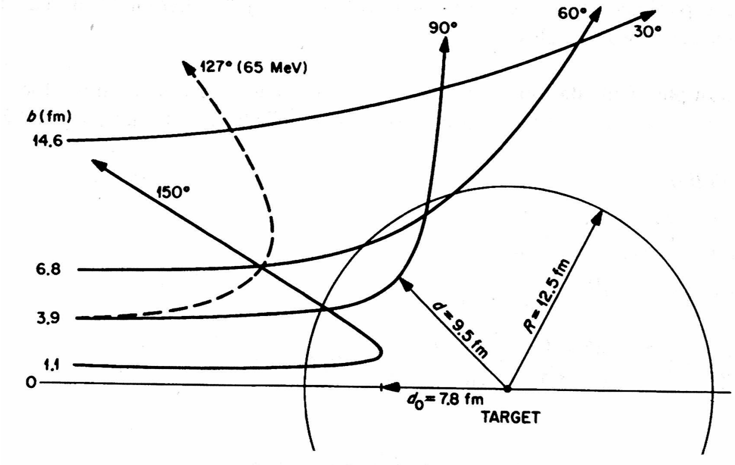
- Since

$$b = \frac{d_0}{2} \frac{1}{\tan\theta} \tag{22}$$

above equations define the relationship between the distance of the closest approach, the impact parameter and the scattering angle.

The distance of the closest approach

^{16}O on ^{208}Pb at 130 MeV



Cross section in Rutherford scattering

- The deviation from the Rutherford cross section with the increasing projectile energy are an evidence for nuclear reactions.