7. Nuclear models

7.1 Fermi gas model

7.2 Shell model

Fermi Gas Model

- Free electron gas: protons and neutrons moving quasi-freely within the nuclear volume
- 2 different potentials wells for protons and neutrons.
- Spherical square well potentials with the same radius

Statistics of the Fermi distribution

Given a volume V the numer of states dn goes like:

$$dn = \frac{4\pi p^2 dp V}{(2\pi\hbar)^3}$$

$$n = \int_0^{p_F} \frac{4\pi p^2 dp V}{(2\pi\hbar)^3} = \frac{V p_F^3}{6\pi^2\hbar^3}$$

If T=0 the nucleus is in the ground state and p_F (Fermi Momentum) is the maximum possible momentum of the ground state.

$$N = 2n = \frac{V p_{F,n}^3}{3\pi^2\hbar^3} \qquad Z = \frac{V p_{F,p}^3}{3\pi^2\hbar^3}$$

N= number of neutrons

Z= number of protons

↓

Spin 1/2 particles

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_0^3 A \quad R_0 = 1.21 fm$$

$$Z = N = A/2 \quad \frac{A}{2} = \frac{4}{3}\frac{\pi R_o^3 A p_F^3}{3\pi^2\hbar^3} \Rightarrow p_F = p_{F,n} = p_{F,p} = \frac{\hbar}{R_0}\left(\frac{9\pi}{8}\right)^{\frac{1}{3}} \approx 250 MeV/c$$

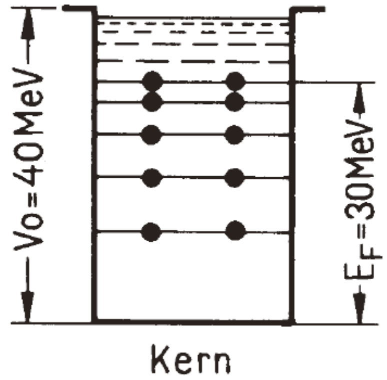
The nucleon moves in the nucleus with a large momentum

Fermi Energy $E_F = \frac{p_F^2}{2m_N} \approx 33\ MeV$

Binding Energy: BE/A= 7-8 MeV

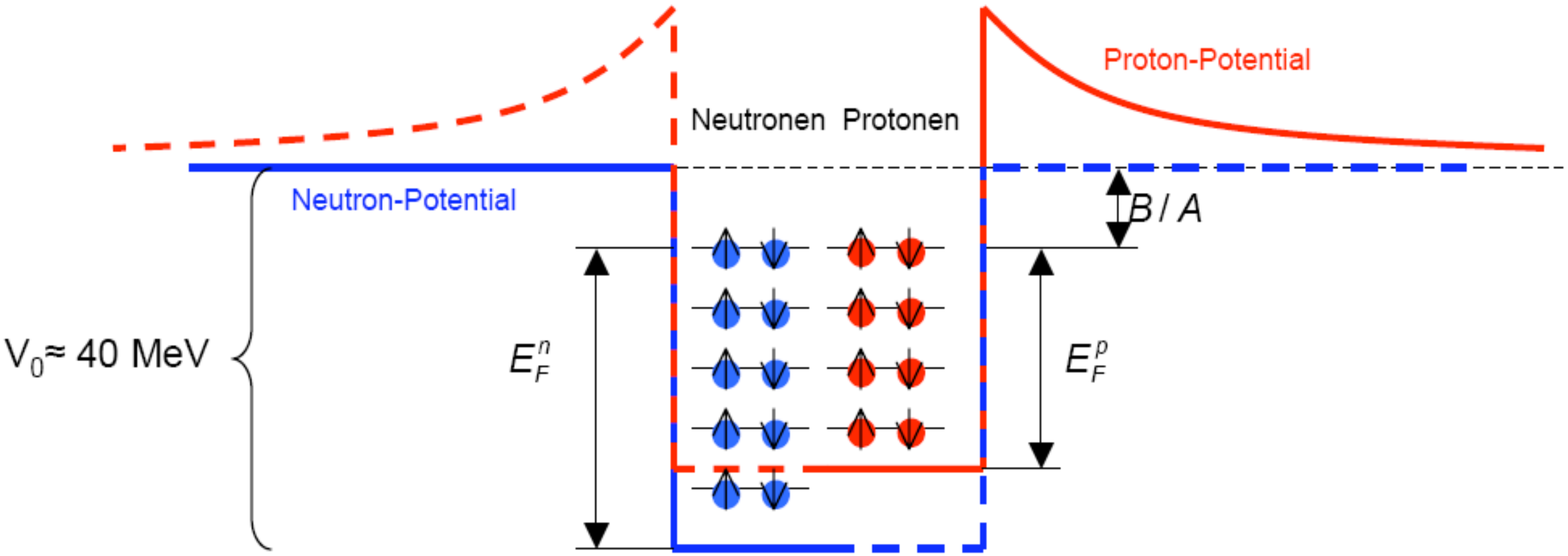
$V_0 = E_F + B/A \sim 40 MeV$

➔Nucleons are rather weakly bound in the nucleus



Potential well in the Fermi-gas model

The neutron potential well is deeper that the proton well because of the missing Coulomb repulsion. The Fermi Energy is the same, otherwise the p-->n decay would happen spontaneously. This implies that they are more neutrons states available and hence N>Z the heavier the nuclei become.



$$< E_{Kin} >= \frac{\int_0^{p_F} E_{Kin} p^2 dp}{\int_0^{p_F} p^2 dp} = \frac{3}{5}\frac{p_F^2}{2m_N} \approx 20\,MeV$$

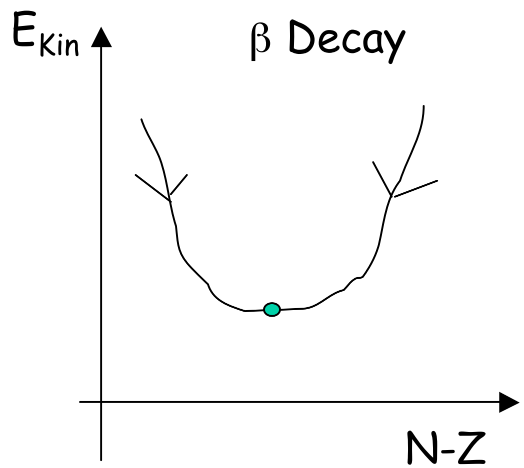
$$E_{Kin}(N,Z) = N < E_{Kin,N} > + Z < E_{Kin,Z} >= \frac{3}{10m_N}(Np_{F,N}^2 + Zp_{FZ}^2)$$

$$p_{FN} = \left(\frac{N \cdot 3\pi^2\hbar^3}{V}\right) = \left(\frac{N \cdot 9\pi^2\hbar^3}{4\pi R_0^3 A}\right)^{\frac{1}{3}}$$

$$p_{FZ} = \left(\frac{Z \cdot 3\pi^2\hbar^3}{V}\right) = \left(\frac{Z \cdot 9\pi^2\hbar^3}{4\pi R_0^3 A}\right)^{\frac{1}{3}}$$

$$E_{Kin} = \frac{3}{19m_N}\frac{\hbar^2}{R_0}\left(\frac{9\pi^2\hbar^3}{4\pi R_0^3}\right)^{\frac{2}{3}}\left(\frac{Z^{\frac{5}{3}} + N^{\frac{5}{3}}}{A^{\frac{2}{3}}}\right)$$

$$E_{Kin} = \frac{\hbar^2}{R_0^2}\frac{3}{10m_N}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\frac{N^{\frac{5}{3}} + Z^{\frac{5}{3}}}{A^{\frac{2}{3}}} = \frac{3}{10m_N}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\left(A + \frac{5}{9}\frac{(N-Z)^2}{A} + \dots\right)$$



Calculation of the state density

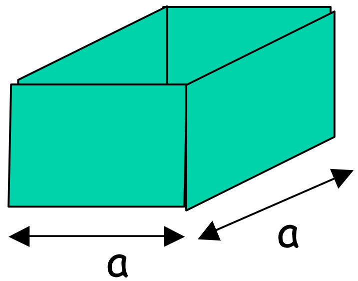
Considering the approximation that the nuclear potential shold have a sharp edge in correspondence of the nuclear radius, one can approximate that to particles trapped in the pot potential

$$-\frac{\hbar^2}{2m}\Delta\psi = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right) = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 X(x)}{\partial x^2} = E_X X(x)..... \quad \psi(r) = X(x)Y(y)Z(z)$$

$$YZE_X X + XZE_y Y + XYE_Z Z = EXYZ$$

$$\frac{\partial^2 X(x)}{\partial x^2} = -kX \quad k = \frac{1}{\hbar}\sqrt{2mE}$$



Volume in which the state density is calculated

$$X_\lambda = A_\lambda e^{ik_\lambda x} + B_\lambda e^{-ik_\lambda x} \quad Boundary\ Conditions: at\ x = y = z = \frac{a}{2} : X = Y = Z = 0$$

$$X_\lambda^+ = \frac{2}{\sqrt{2a}}\cos k_\lambda^+ x \quad X_\lambda^- = \frac{2i}{\sqrt{2a}}\sin k_\lambda^- x \quad k_\lambda^+ = \frac{\pi\lambda^+}{a}, \lambda = 1,3,5.. \quad k_\lambda^- = \frac{\pi\lambda^-}{a}, \lambda = 0,2,4...$$

$$EX = \frac{\hbar^2}{2m}k_\lambda^2 = \frac{1}{2m}\left(\frac{\pi\lambda\hbar}{a}\right)^2$$

$$E = \frac{1}{2m}\left(\frac{\pi\hbar}{a}\right)^2(\lambda_x^2 + \lambda_y^2 + \lambda_z^2) \quad p^2 = 2mE = \left(\frac{\pi\hbar}{a}\right)^2(\lambda_x^2 + \lambda_y^2 + \lambda_z^2)$$

$$\Omega \approx \lambda^3 \approx a^3 p^3$$

We count how many states we have in a spherical volume of radius between ρ and ρ+δρ

$$\rho^2 = \lambda_x^2 + \lambda_y^2 + \lambda_z^2 \qquad d\Omega = 4\pi\rho^2\delta\rho$$

$$dn = \frac{1}{8}d\Omega = \frac{\pi}{2}\rho^2\delta\rho \qquad \rho = \frac{a}{\pi\hbar}p \qquad d\rho = \frac{a}{\pi\hbar}dp$$

$$dn = \frac{\pi}{2}\frac{a^2}{\pi^2\hbar^2}p^2\frac{a}{\pi\hbar}dp = \frac{a^3}{2\pi^2\hbar^3}p^2dp = \frac{4\pi p^2 dp\, v}{(2\pi\hbar)^3}$$

Momentum volume

Space volume

Phase-space occupied by one state

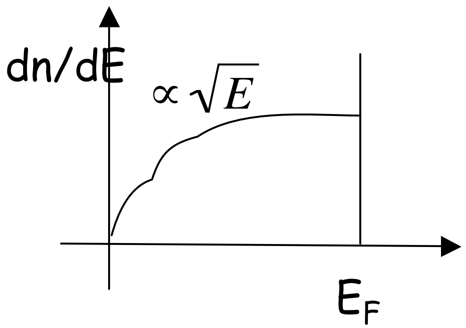
$$p^2 = 2mE \qquad p^2dp = \sqrt{2m^3E}\,dE$$

$$dn = m^{\frac{3}{2}}(\sqrt{2}\pi^2\hbar^3)^{-1}v\,dE\sqrt{E} = C_1\sqrt{E}\,dE$$

Number of spin 1/2 particles which sit in the well:

$$n = \int_0^{E_F} 2\frac{dn}{dE}dE = 2C_1 v\int_0^{E_F}\sqrt{E}\,dE = (\sqrt{8}m^{\frac{3}{2}})(3\pi^2\hbar^3)^{-1}v\,E_F^{\frac{2}{3}}$$

E_F= maximal energy of the particle in the well



$$E_F = \left(\frac{1}{2m}\right)3^{\frac{2}{3}}\pi^{\frac{4}{3}}\hbar^2\left(\frac{n}{v}\right)^{2/3} \approx 30\,MeV$$

$$in \quad Cu\ atoms: \frac{n}{v} = 6\cdot 10^{23}\cdot\frac{9}{64} = 8\cdot 10^{22}cm^{-3} \Rightarrow E_F \approx 7eV$$

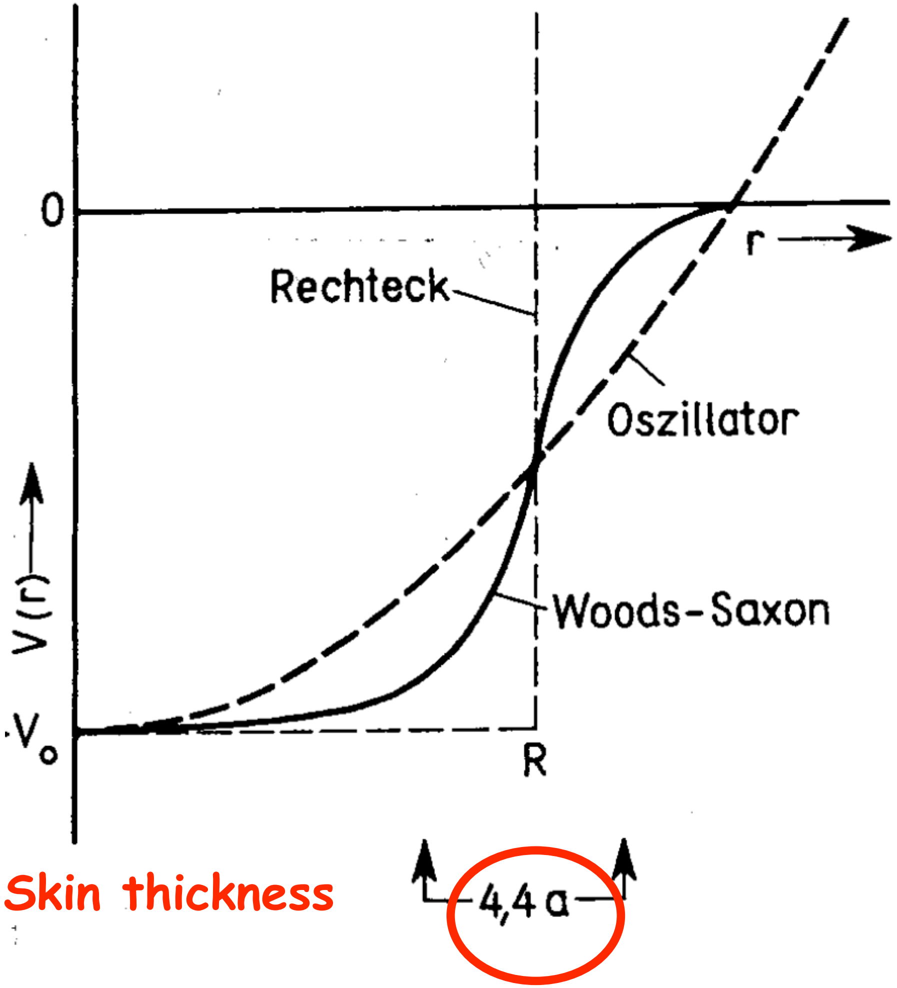
Hierarchy of energy eigenstates of the harmonic oscillator potential

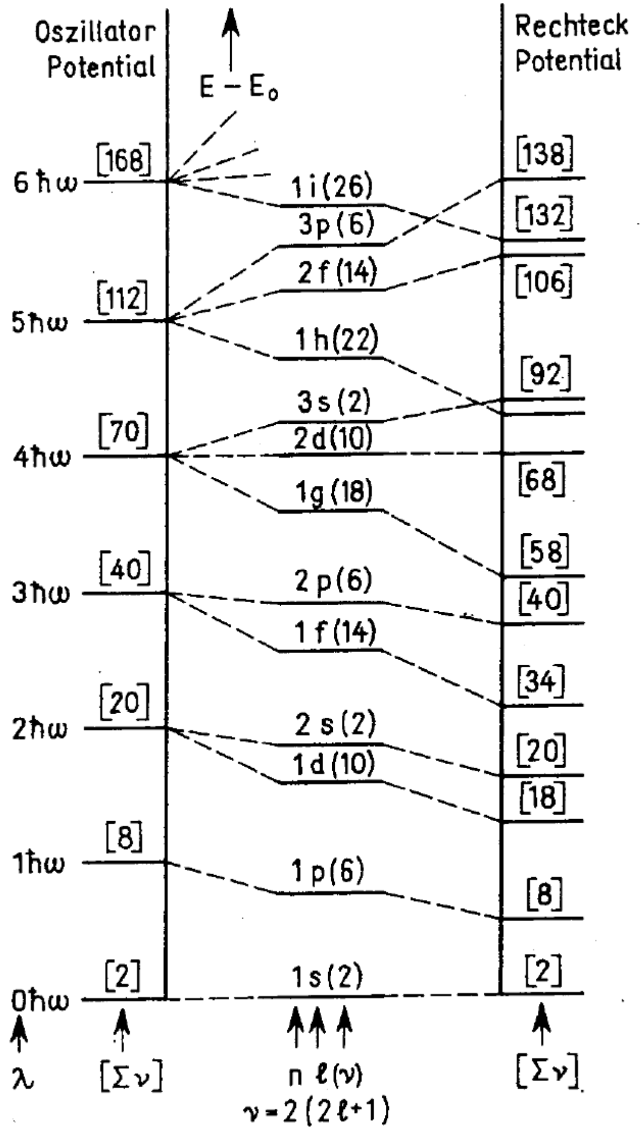
N	0	1	2	2	3	3	4	4	4	...
$n\ell$	1s	1p	1d	2s	1f	2p	1g	2d	3s	...
Degeneracy	2	6	10	2	14	6	18	10	2	...
States with $E \leq E_{n\ell}$	2	8	18	20	34	40	58	68	70	...

$$N = 2(n-1) + l$$

$$E = h\nu(N + 3/2)$$

Shell Model Potential





Spin Orbit Coupling

It changes the hierarchy of the energy levels:

$$V(r) = V_{central}(r) + V_{ls}(r)\frac{<\vec{l}\vec{s}>}{\hbar^2}$$

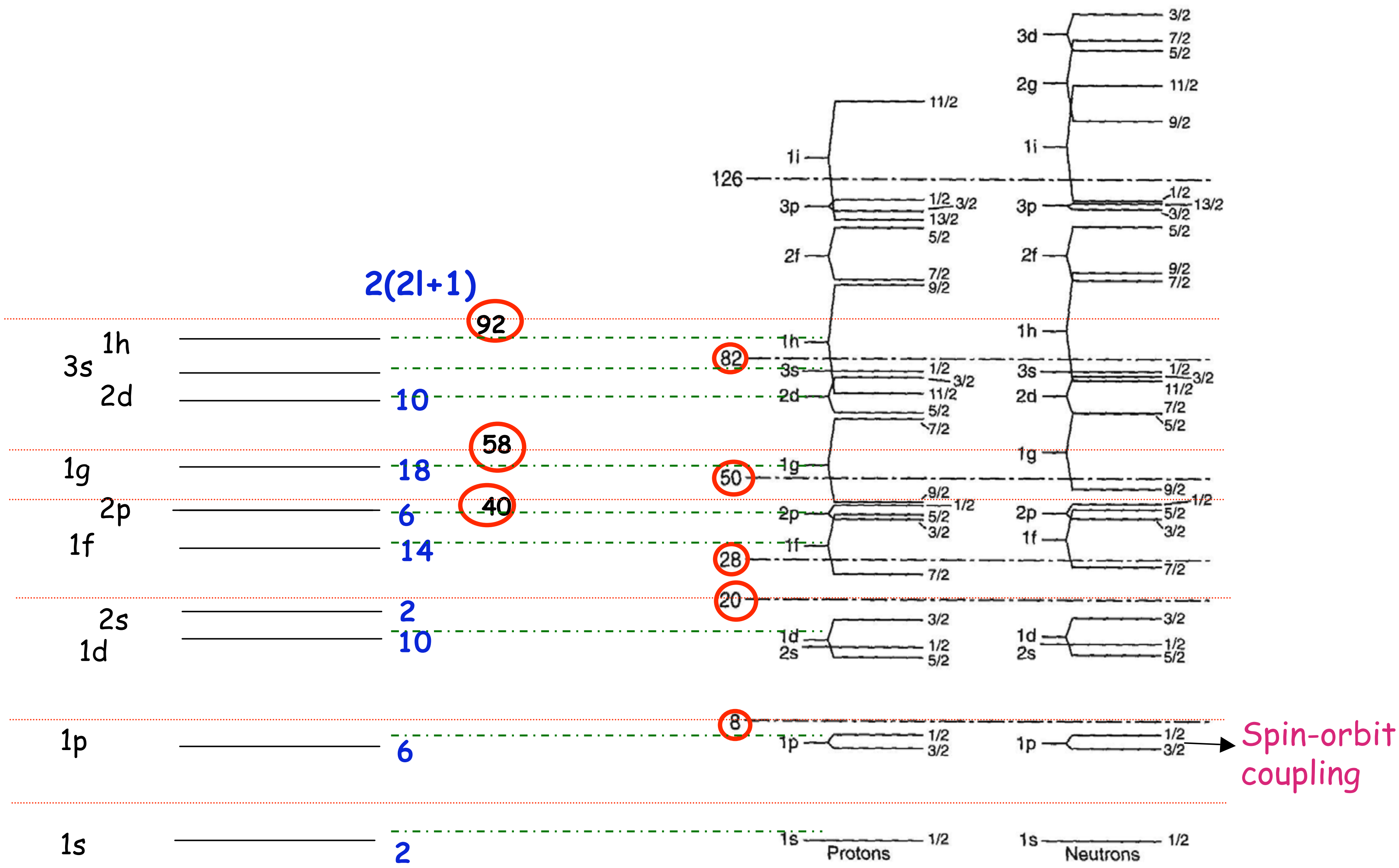
$$\frac{<\vec{l}\vec{s}>}{\hbar^2} = \frac{1}{2}(j(j+1) - l(l+1) - s(s+1)) = \begin{cases} l/2 & j = l+1/2 \\ -(l+1)/2 & j = l-1/2 \end{cases}$$

Moving below
Moving above

$$\Delta E_{ls} = \frac{2(l+1)}{2} < V_{ls}(r) >$$

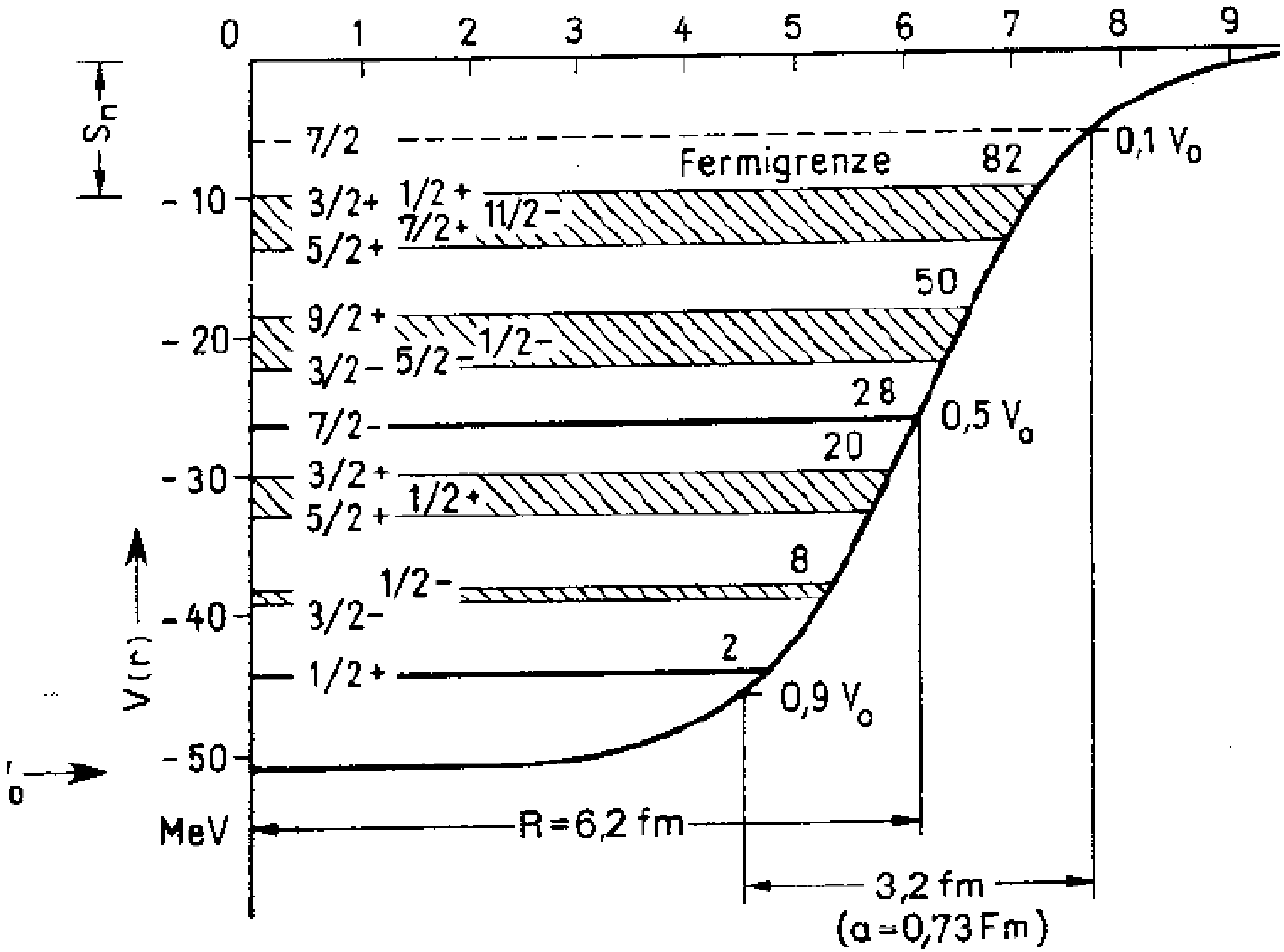
The energy levels transform into n_{lj} levels.

Saxon-Wood Potential Shell model

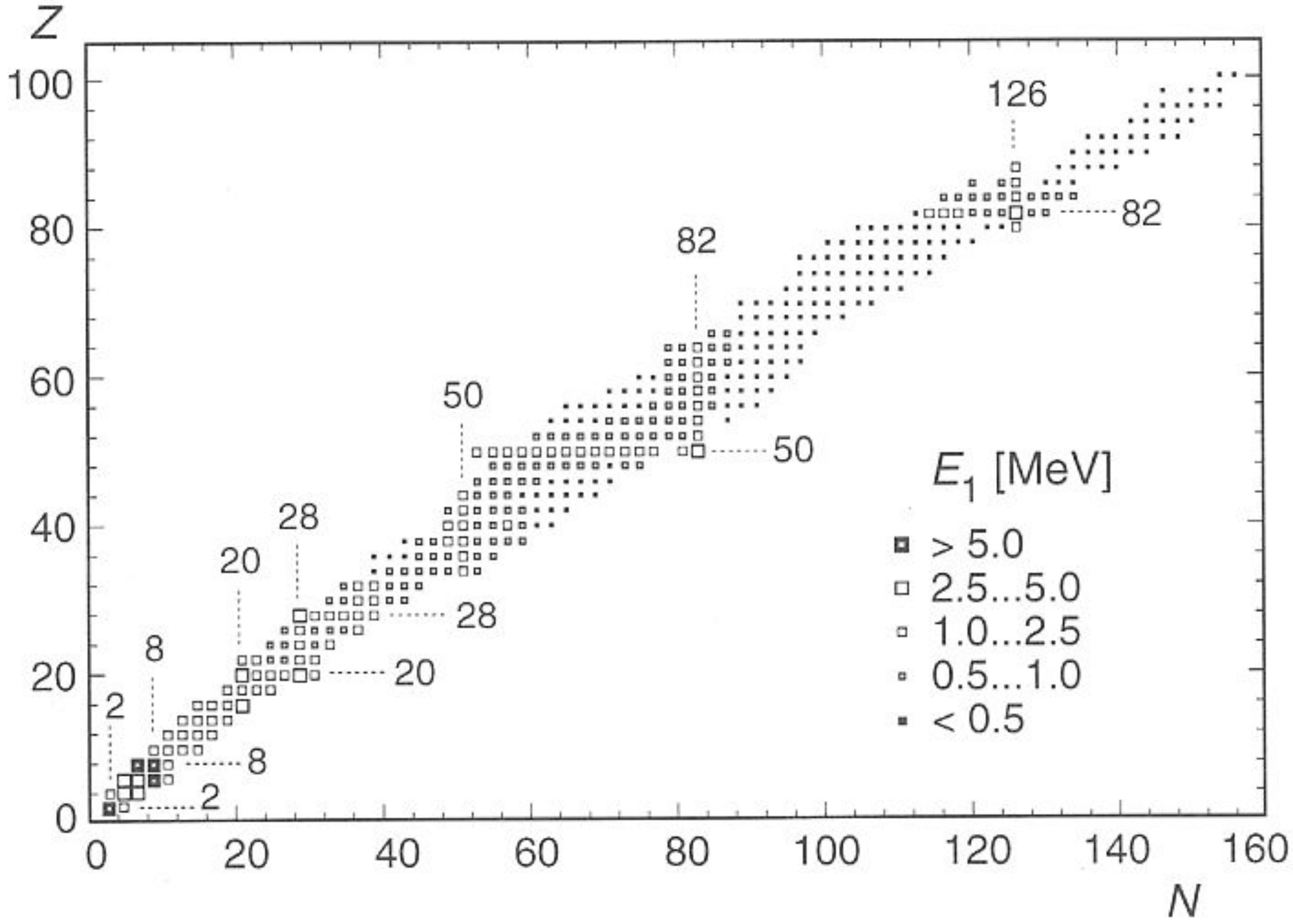


Single particle level calculated in the shell model.

Single-Particle-Spectrum of the Wood-Saxon Potential for a heavy nucleus

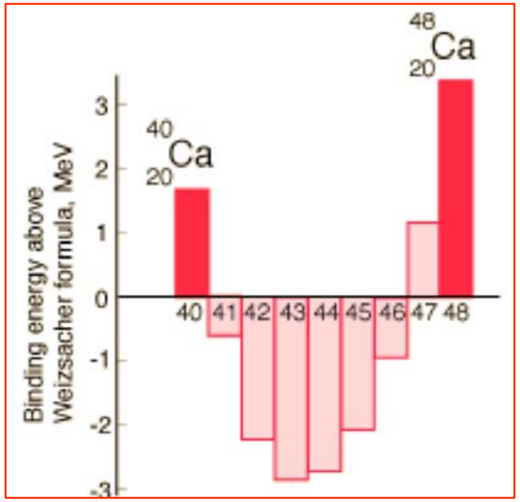


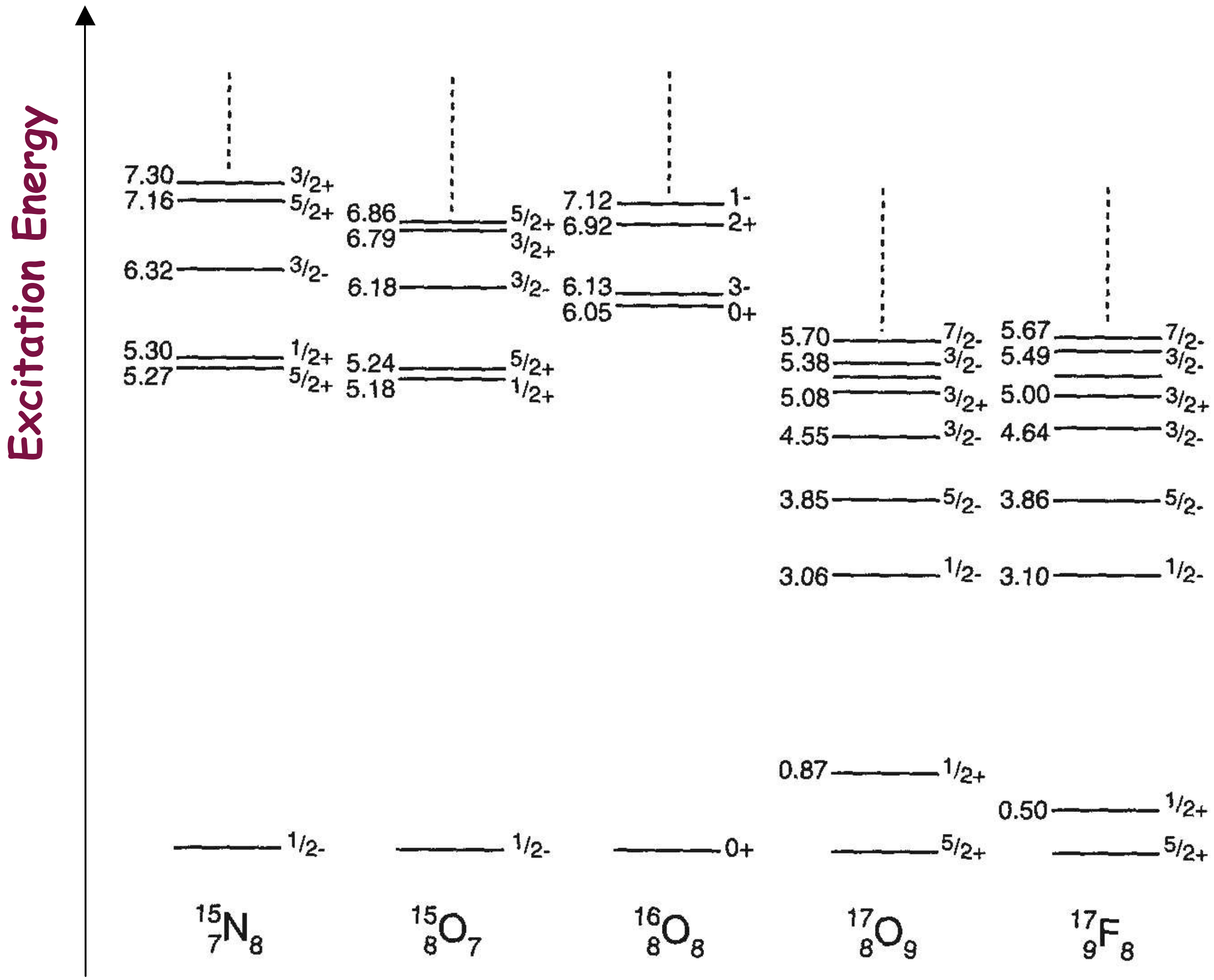
Energy of first excited nuclear level in gg-nuclei



Energy levels of some nuclei

$^{4}_{2}He$ $^{16}_{8}O$ $^{40}_{20}Ca$ $^{48}_{20}Ca$ $^{208}_{82}Pb$





Nuclear Magnetic Moments in Shell Model

$$\mu_{nucl} = \mu_N \sum_{i=1}^{A} (g_l \vec{l}_i + g_s \vec{s}_i)/\hbar$$

$$g_l = \begin{cases} 1 & for\ p \\ 0 & for\ n \end{cases} \qquad g_s = \begin{cases} 5.58 & for\ p \\ -3.83 & for\ n \end{cases}$$

$$< \mu_{nucl} >= \mu_N \sum_{i=1}^{A} < \psi_{nucl} \left| g_l \vec{l}_i + g_s \vec{s}_i \right| \psi_{nucl} > /\hbar$$

Wigner-Eckart Theorem: The expectation value of any vector operator of a system is equal to the projection onto ist angular momentum

$$< \mu_{nucl} >= g_{nucl} \mu_N \frac{< \vec{J} >}{\hbar} \qquad g_{nucl} = \sum_{i=1}^{A} \frac{< JM_J \left| g_l \vec{l}_i + g_s \vec{s}_i \right| JM_J >}{< JM_J \left| J^2 \right| JM_J >}$$

In the case of a single nucleon in addition to the close shell the angular momentum of the closed shell nuclei couples to 0. The nuclear magnetic Momentum of is equal to the nuclear magnetic mom. of the valence nucleon.

$$g_{nucl} = \frac{g_l (j(j+1) + l(l+1) - s(s+1)) + gs(j(j+1) + s(s+1) - l(l+1))}{2j(j+1)}$$

$$\frac{\mu_{nucl}}{\mu_N} = g_{nucl} J = \left(g_l \pm \frac{g_s - g_l}{2l+1} \right) \qquad J = l \pm 1/2$$

Magnetic moments:

Shell model calculations and experimental values

Nucleus	State	J^P	μ/μ_N Model	μ/μ_N Expt.
^{15}N	$p\text{-}1p_{1/2}^{-1}$	$1/2^-$	-0.264	-0.283
^{15}O	$n\text{-}1p_{1/2}^{-1}$	$1/2^-$	$+0.638$	$+0.719$
^{17}O	$n\text{-}1d_{5/2}$	$5/2^+$	-1.913	-1.894
^{17}F	$p\text{-}1d_{5/2}$	$5/2^+$	$+4.722$	$+4.793$