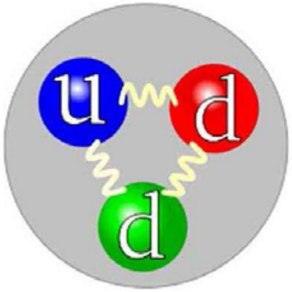
Nuclear Shell Model

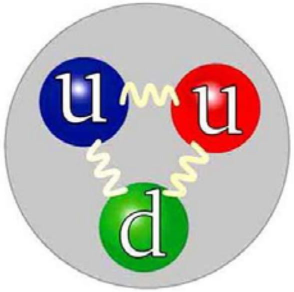
The nucleon building blocks...

Name	up quark (u)	down quark (d)
mass (MeV)	1.7 – 3.1	4.1 – 5.7
charge (e)	+2/3	-1/3
spin	1/2	1/2

The nuclear building blocks...

Name	neutron	proton
mass (MeV)	939.565378(21)	938.272046(21)
charge (e)	0	1
constituents	$2d + 1u$	$2u + 1d$
I^π	$1/2^+$	$1/2^+$

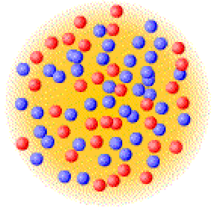




➢ Complexity out of simplicity – Microscopic

How the world, with all its apparent complexity and diversity can be constructed out of a few elementary building blocks and their interactions

individual excitations of nucleons

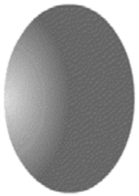


➢ Simplicity out of complexity – Macroscopic

How the world of complex systems can display such remarkable regularity and simplicity

vibration

rotation



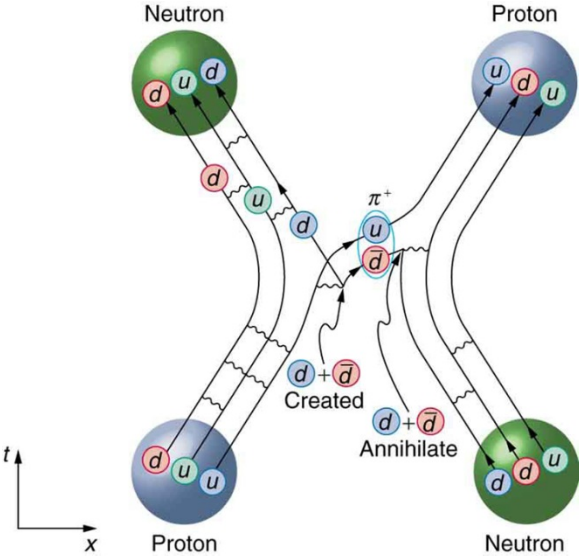
fission

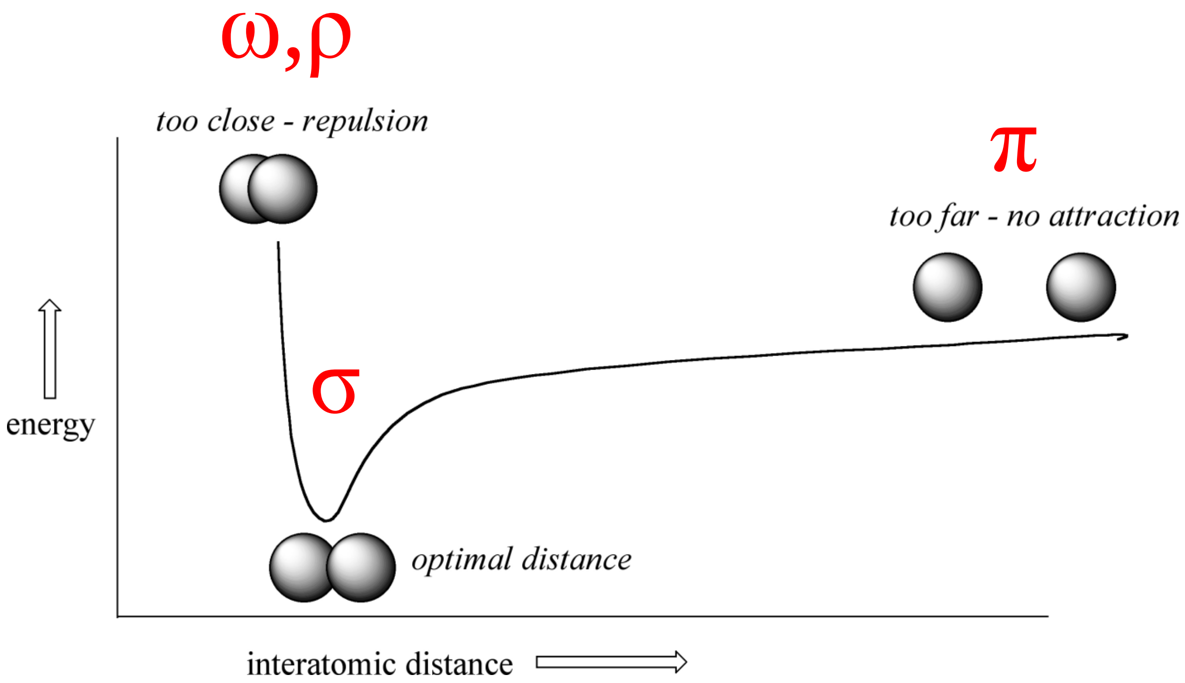
The nuclear force

The nuclear force is short-range, but does not allow for compression of nuclear matter.

Yukawa – potential:

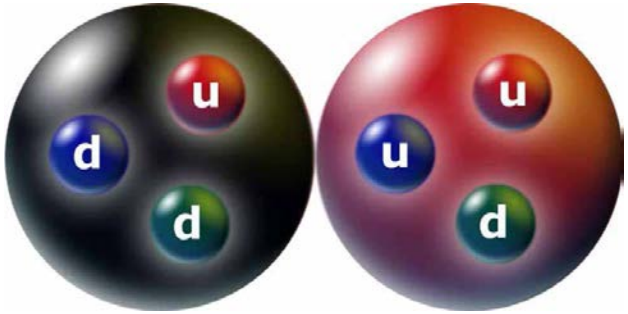
$$V_0(r) = g_s \cdot \frac{1}{r} \cdot e^{-\left(\frac{m_\pi c}{\hbar}\right) \cdot r}$$



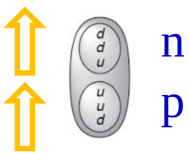


$m(\pi) \approx 140\ MeV/c^2$
$m(\sigma) \approx 500\text{-}600\ MeV/c^2$
$m(\omega) \approx 784\ MeV/c^2$

The deuteron



mass (MeV/c^2)	1875.61	
charge (e)	1	
I^{π}	1^{+}	
binding energy (MeV)	2.2245	
magnetic moment (μ_N)	0.8574	$0.8798\ \mu_N = \mu_S\ (^{2}_{1}H)$ → the deuteron can not be a pure ***s*** state! ~ 96% s and 4% d.
quadrupole moment (b)	0.0029	not spherical consistent with s/d-ratio = 96/4



The deuteron is an ideal candidate for tests of our basic understanding of nuclear physics

Structure of the nuclear force

Structure of the nuclear force is more complex than e.g. Coulomb force. It results from its structure as residual interaction of the colorless nucleons.

central force $V_0(r)$

results from deuteron properties (96% $^3S_1$ state) $^{2S+1}L_J$

spin dependent central force

results from neutron-proton scattering (spin-spin interaction)

not central tensor force

results from deuteron properties (4% $^3D_1$ state) $^{2S+1}L_J$

spin-orbit ($\ell \cdot s$) term

results from scattering of polarized protons (left/right asymmetry)

$$\begin{aligned} V(r) = {} & V_0(r) && \text{central potential} \\ & + V_{ss}(r) \cdot \vec{s_1} \cdot \vec{s_2} \cdot \frac{1}{\hbar^2} && \text{spin-spin interaction} \\ & + V_T(r) \cdot \frac{3}{\hbar^2} \frac{(\vec{s_1} \cdot \vec{x})(\vec{s_2} \cdot \vec{x})}{r^2} - \vec{s_1} \cdot \vec{s_2} && \text{tensor force} \\ & + V_{\ell s}(r) \cdot (\vec{s_1} + \vec{s_2}) \cdot \vec{\ell} \cdot \frac{1}{\hbar^2} && \text{spin-orbit interaction} \end{aligned}$$

Structure of the nuclear force

❖ spin-spin force:

$$\sim V_{ss}(r)\cdot\vec{s_1}\cdot\vec{s_2}\,/\hbar^2$$

different eigenvalues for
triplet and singlet states

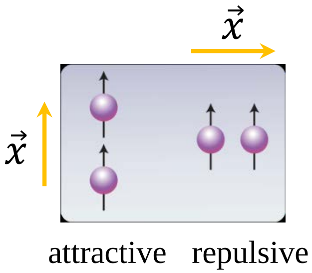
$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\right)$ s = 0, ℓ = 1

$|\uparrow\uparrow\rangle$ $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\right)$ $|\downarrow\downarrow\rangle$ s = 1, ℓ = 0

❖ tensor force:

$$\sim V_T(r)\cdot\frac{3}{\hbar^2}\frac{(\vec{s_1}\cdot\vec{x})(\vec{s_2}\cdot\vec{x})}{r^2}-\vec{s_1}\cdot\vec{s_2}$$

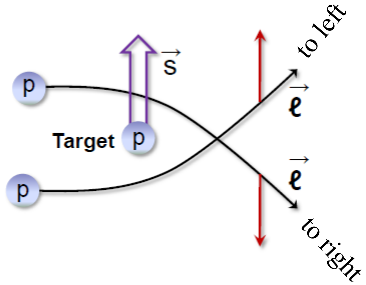
small deformation of deuterium
maximum magnetic dipole moments



❖ ℓ·s coupling:

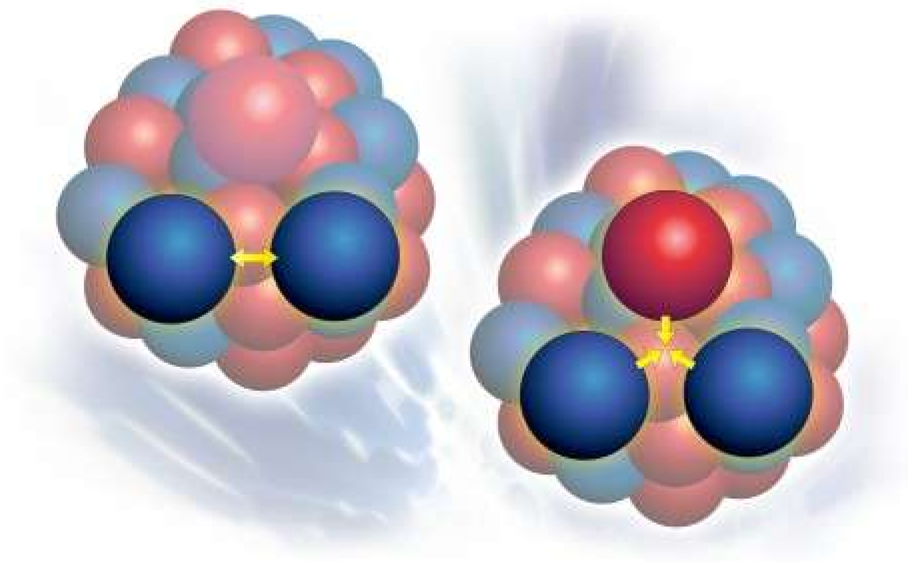
$$\sim V_{\ell s}(r)\cdot(\vec{\ell}\cdot\vec{s})$$

scattering of protons on polarized protons
asymmetry of counting rates
- left scattering: $\vec{\ell}\cdot\vec{s}>0$
- right scattering: $\vec{\ell}\cdot\vec{s}<0$

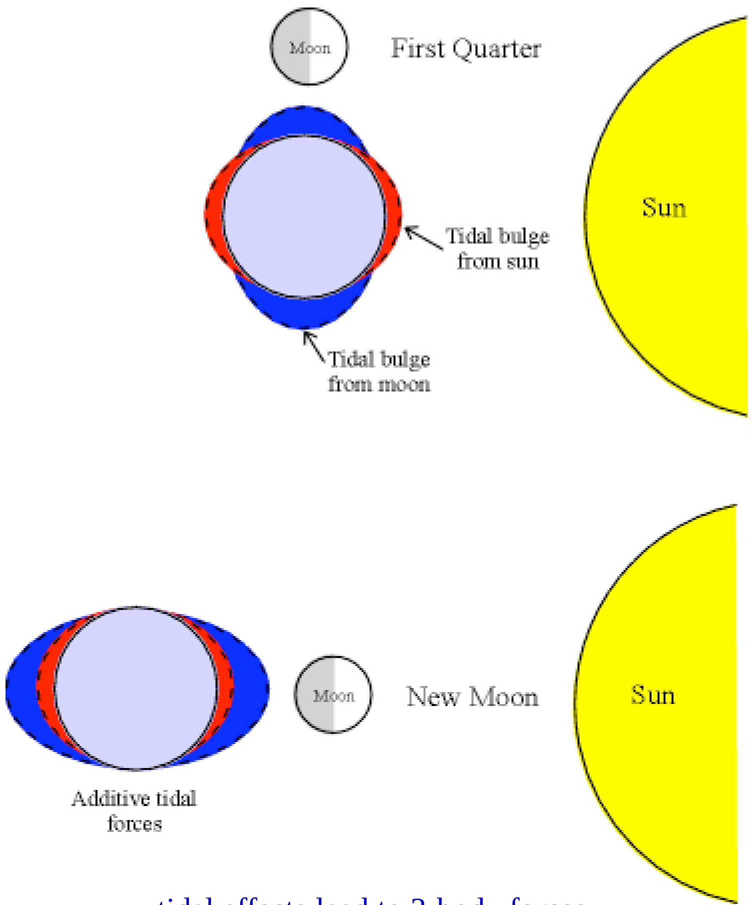


ℓ·s coupling:
- no net contribution in the center of nucleus
- radial dependence at the surface of the nucleus $V_{\ell s}(r)\propto\frac{1}{r}\cdot\frac{d\rho}{dr}$

Many-body forces



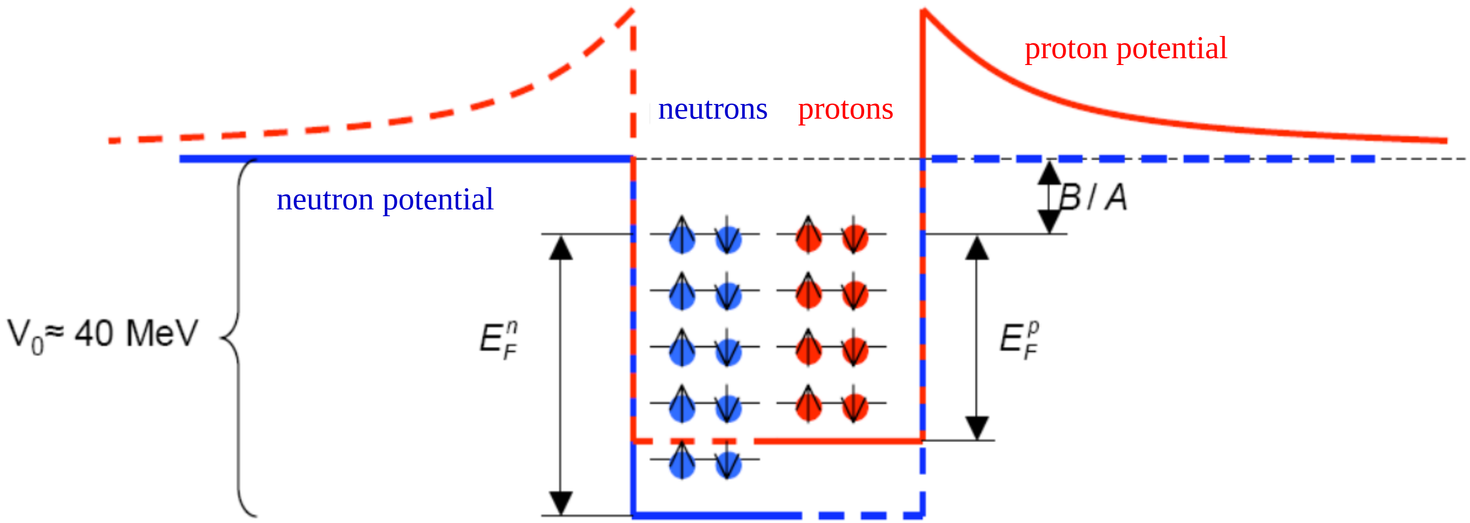
The force on one nucleon does not only depend on the position of the other nucleons, but also on the distance between the other nucleons! These are called many-body forces.



tidal effects lead to 3-body forces in earth-sun-moon system

<u>Remember:</u> Nucleons are finite-mass composite particles, can be excited to resonances. Dominant contribution Δ(1232 MeV)

The Fermi gas model



- The Fermi gas model assumes that protons and neutrons are moving freely within the nuclear volume. They are distinguishable fermions (s = ½) filling two separate potential wells obeying the Pauli principle (↑↓-pair).
- The model assumes that all fermions occupy the lowest energy states available to them to the highest occupied state (Fermi energy), and that there is no excitation across the Fermi energy (i.e. zero temperature).
- The Fermi energy is common for protons and neutrons in stable nuclei.
- If the Fermi energy for protons and neutrons are different then the β-decay transforms one type of nucleons into the other until the common Fermi energy (stability) is reached.

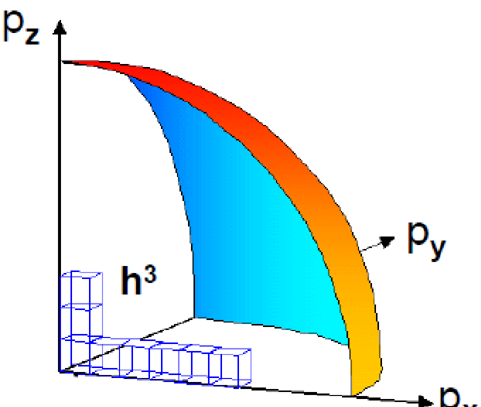
Number of nucleon states

Heisenberg Uncertainty Principle: $\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar$

The volume of one particle in phase space: $2\pi \cdot \hbar$

The number of nucleon states in a volume V:

$$n = \frac{\iint d^3r\, d^3p}{(2\pi \cdot \hbar)^3} = \frac{V \cdot 4\pi \int_0^{p_{max}} p^2\, dp}{(2\pi \cdot \hbar)^3}$$



states in phase space

At temperature T = 0, i.e. for the nucleus in its ground state, the lowest states will be filled up to the maximum momentum, called the Fermi momentum p_F. The number of these states follows from integration from 0 to $p_{max} = p_F$.

$$n = \frac{V \cdot 4\pi \int_0^{p_F} p^2 dp}{(2\pi \cdot \hbar)^3} = \frac{V \cdot 4\pi \cdot p_F^3}{(2\pi \cdot \hbar)^3 \cdot 3} \quad \rightarrow \quad \boxed{n = \frac{V \cdot p_F^3}{6\pi^2 \hbar^3}}$$

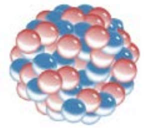
Since an energy state can contain two fermions of the same species, we can have

$$neutrons: \quad N = \frac{V \cdot (p_F^n)^3}{3\pi^2 \hbar^3} \qquad\qquad protons: \quad Z = \frac{V \cdot \left(p_F^p\right)^3}{3\pi^2 \hbar^3}$$

p_F^n is the Fermi momentum for neutrons, p_F^p for protons

Fermi momentum

Use $R = r_0 \cdot A^{1/3}\ fm$

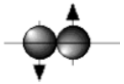


$$V = \frac{4\pi}{3} R^3 = \frac{4\pi}{3} r_0^3 \cdot A$$

The density of nucleons in a nucleus = number of nucleons in a volume V:

$$n = 2 \cdot \frac{V \cdot p_F^3}{6\pi^2 \hbar^3} = 2 \cdot \frac{4\pi}{3} r_0^3 \cdot A \cdot \frac{p_F^3}{6\pi^2 \hbar^3} = \frac{4A}{9\pi} \frac{r_0^3 \cdot p_F^3}{\hbar^3}$$

two spin states



Fermi momentum p_F:

$$p_F = \left(\frac{6\pi^2 \hbar^3 n}{2V}\right)^{1/3} = \left(\frac{9\pi\hbar^3}{4A} \frac{n}{r_0^3}\right)^{1/3} = \left(\frac{9\pi \cdot n}{4A}\right)^{1/3} \cdot \frac{\hbar}{r_0}$$

After assuming that the proton and neutron potential wells have the same radius, we find for a nucleus with n = Z = N = A/2 the Fermi momentum p_F.

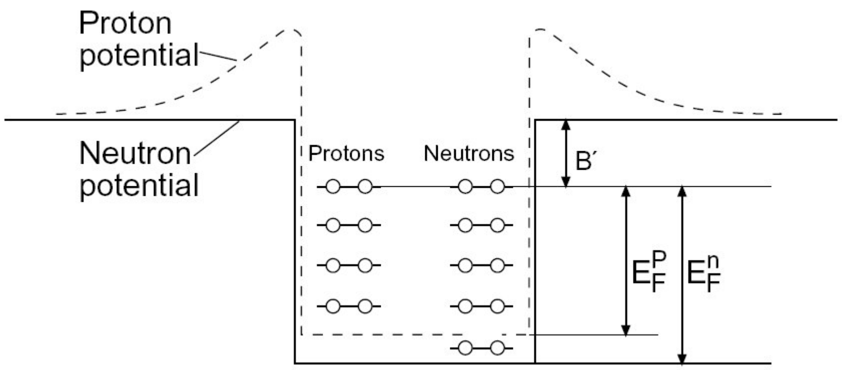
$$p_F = p_F^n = p_F^p = \left(\frac{9\pi}{8}\right)^{1/3} \cdot \frac{\hbar}{r_0} \approx 250\ MeV/c$$

The nucleons move freely inside the nucleus with large momenta

Fermi energy: $E_F = \frac{p_F^2}{2m_N} \approx 33\ MeV$

m_N = 938 MeV/c^2 – the nucleon mass

Nucleon potential



The difference B´ between the top of the well and the Fermi level is the average binding energy per nucleon B/A = 7 – 8 MeV.

→ The depth of the potential V_0 and the Fermi energy are independent of the mass number A:

$$V_0 = E_F + B' \approx 40\ MeV$$

Heavy nuclei have a surplus of neutrons. Since the Fermi level of the protons and neutrons in a stable nucleus have to be equal (otherwise the nucleus would enter a more energetically favorable state through β-decay) this implies that the depth of the potential well as it is experienced by the neutron gas has to be larger than of the proton gas.

Protons are therefore on average less strongly bound in nuclei than neutrons. This may be understood as a consequence of the Coulomb repulsion of the charged protons and leads to an extra term in the potential:

$$V_C = (Z-1)\frac{\alpha \cdot \hbar c}{R}$$

Protonen: 33MeV + 7MeV, Neutronen: 43MeV + 7 MeV

Assumption: neutron star as cold neutron gas with constant density

- 1.5 sun masses: M = $3 \cdot 10^{30}$ kg ($m_N = 1.67 \cdot 10^{-27}$ kg), number of neutrons: n = $1.8 \cdot 10^{57}$

Fermi momentum p_F for cold neutron gas:

$$p_F = \left(\frac{9\pi \cdot n}{4}\right)^{1/3} \cdot \frac{\hbar}{R}$$

R is the radius of the neutron star

Average kinetic energy per neutron:

$$\left\langle E_{kin}/N \right\rangle = \frac{3}{5} \cdot \frac{p_F^2}{2m_N} = \left(\frac{9\pi \cdot n}{4}\right)^{2/3} \cdot \frac{3\hbar^2}{10 \cdot m_N} \cdot \frac{1}{R^2} = \frac{C}{R^2}$$

Gravitational energy of a star with constant density has an average potential energy per neutron:

$$\left\langle E_{pot}/N \right\rangle = -\frac{3}{5} \cdot \frac{G \cdot n \cdot m_n^2}{R} = -\frac{D}{R}$$

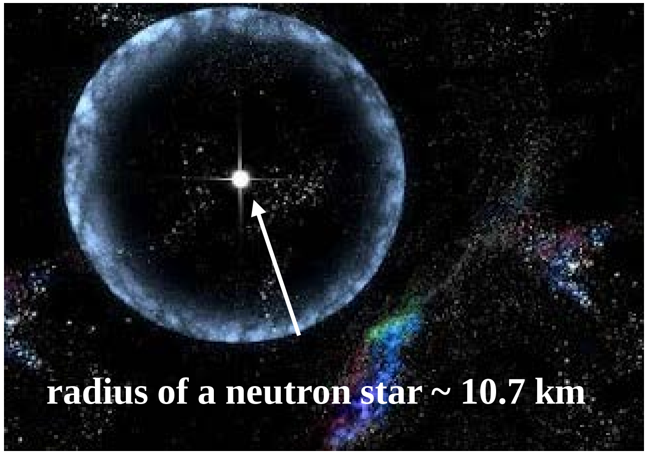
$$G = 6.67 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$$

Minimum total energy per neutron:

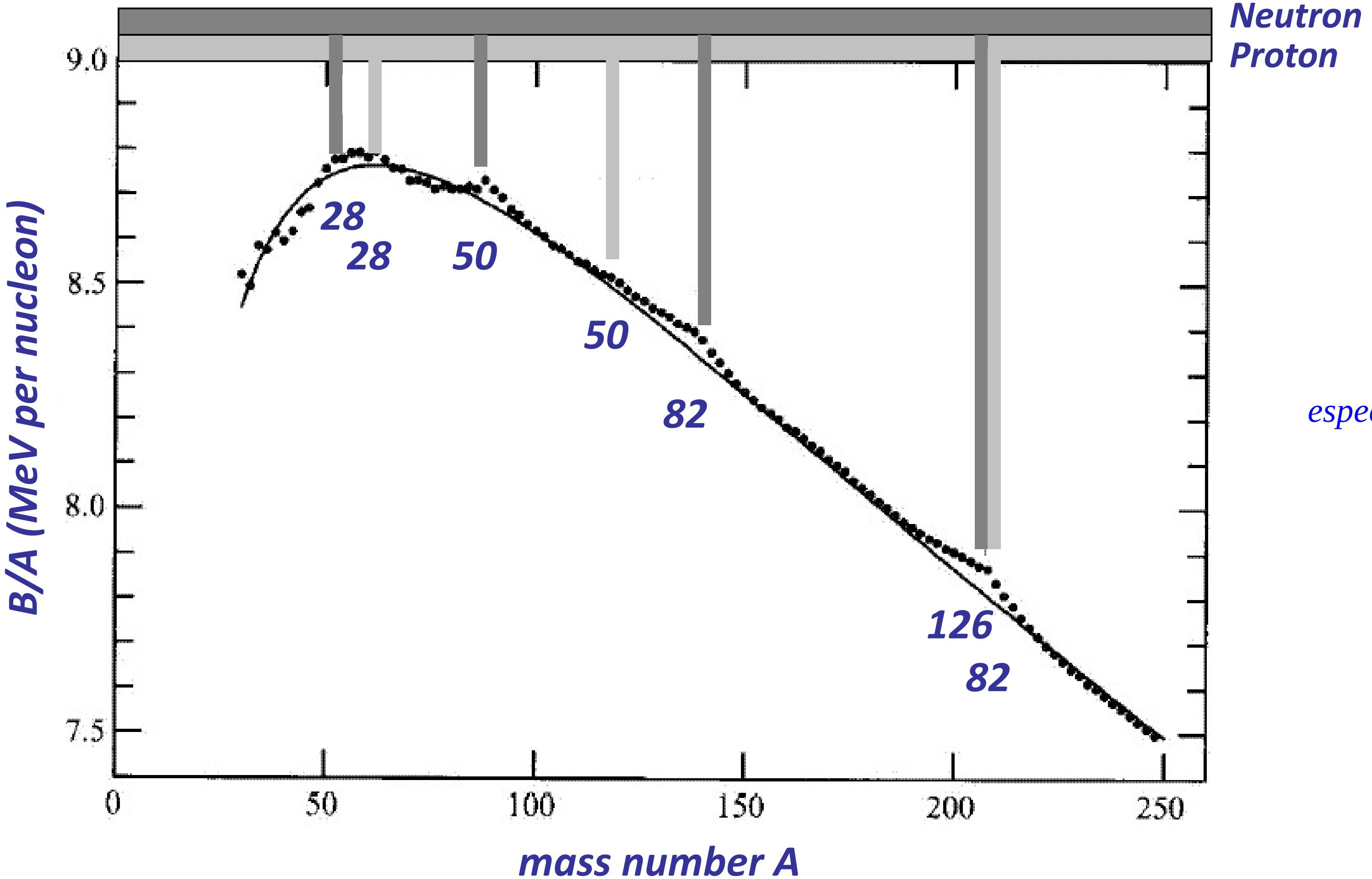
$$\frac{d}{dR}\langle E/N \rangle = \frac{d}{dR}\left[\langle E_{kin}/N \rangle + \langle E_{pot}/N \rangle\right] = 0$$

$$\frac{d}{dR}\left[\frac{C}{R^2} - \frac{D}{R}\right] = -\frac{2C}{R^3} + \frac{D}{R^2} = 0$$

$$R = \frac{2C}{D} \quad \rightarrow \quad R = \frac{\hbar^2 \cdot (9\pi/4)^{2/3}}{G \cdot m_N^3 \cdot n^{1/3}}$$



Deviations from the Bethe-Weizsäcker mass formula:



especially stable:

$^{4}_{2}He_{2}$

$^{16}_{8}O_{8}$

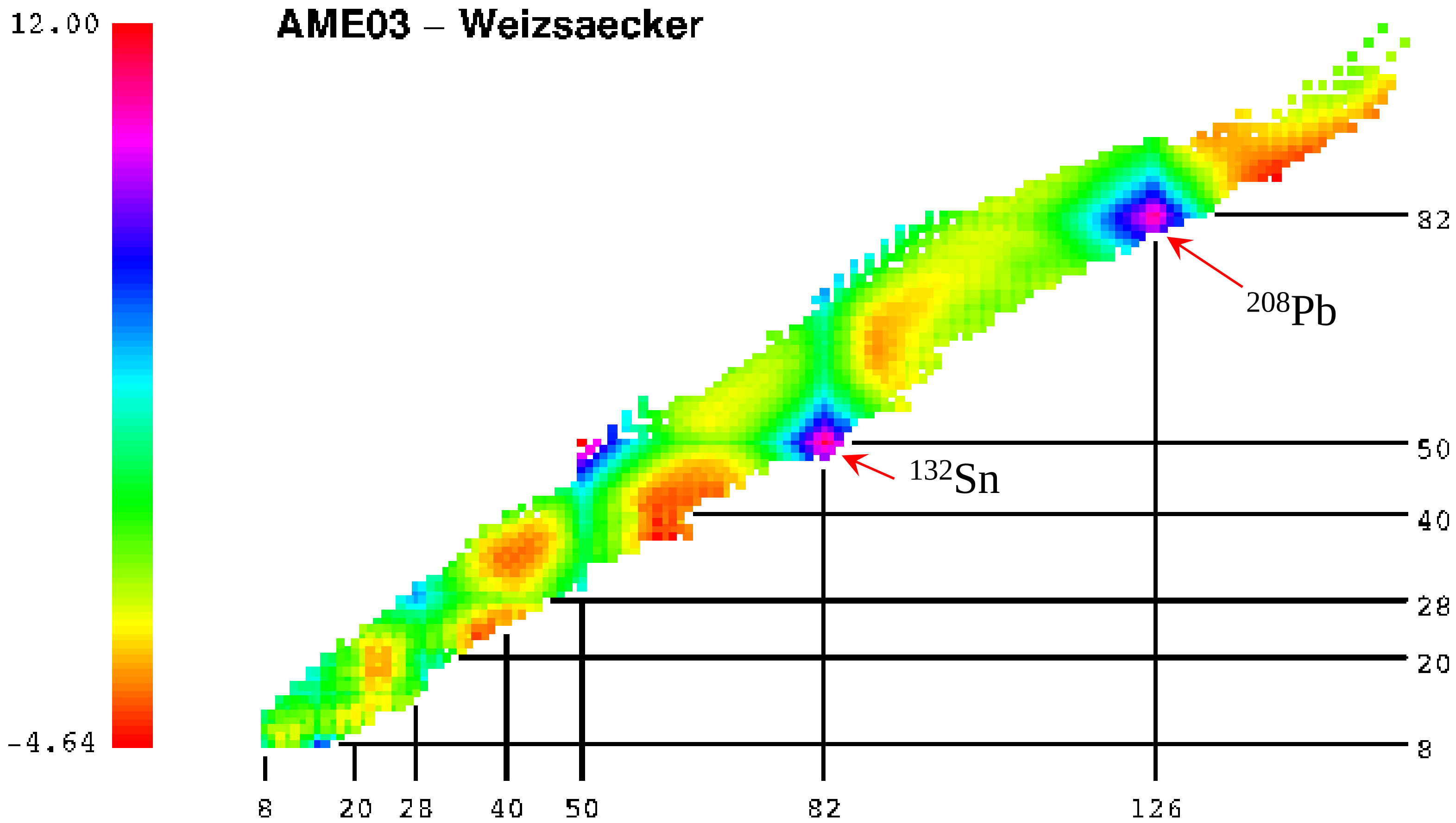
$^{40}_{20}Ca_{20}$

$^{48}_{20}Ca_{28}$

$^{208}_{82}Pb_{126}$

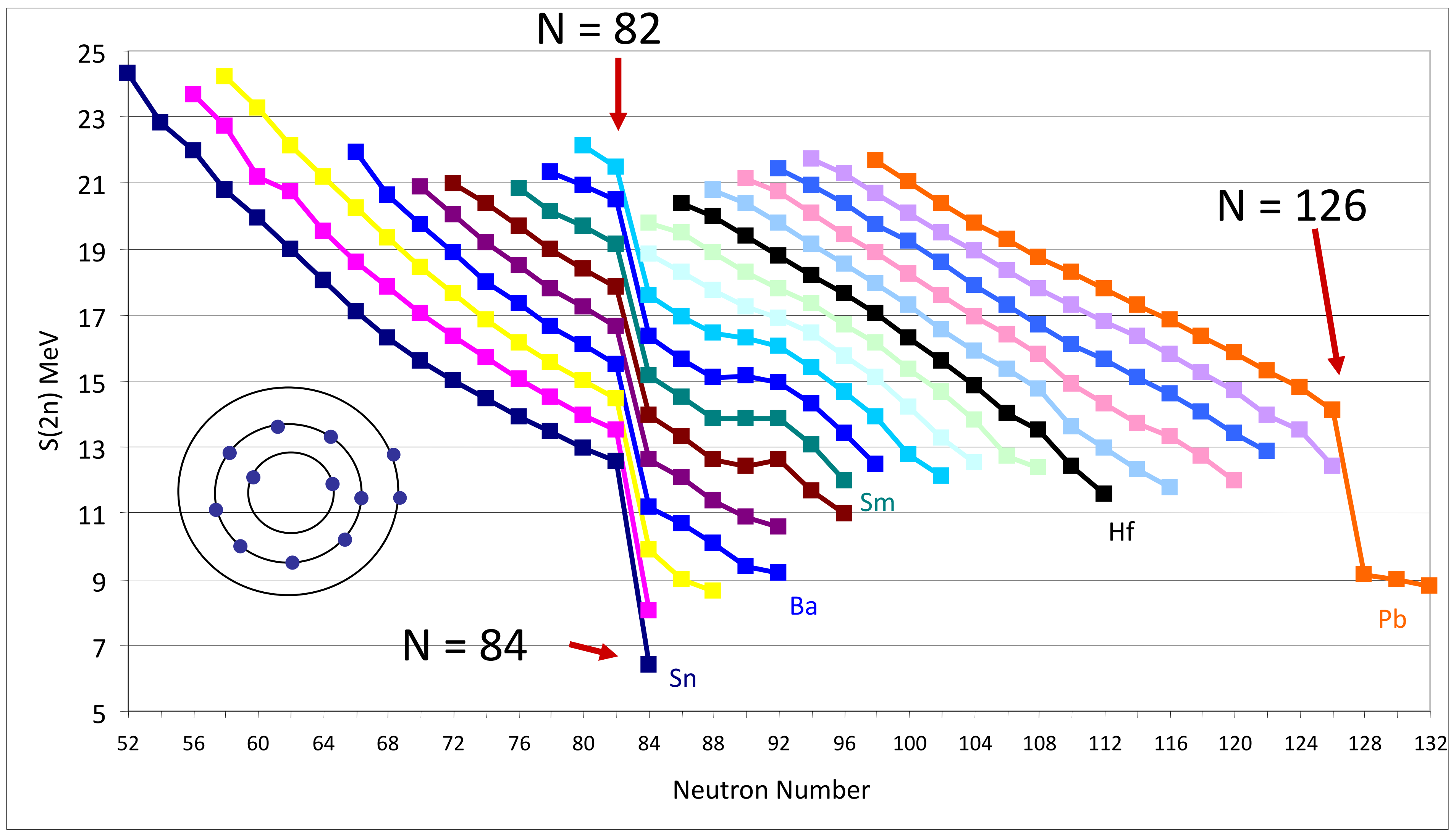
Shell structure in nuclei

- *deviations from the Bethe-Weizsäcker mass formula:* ***large binding energies***

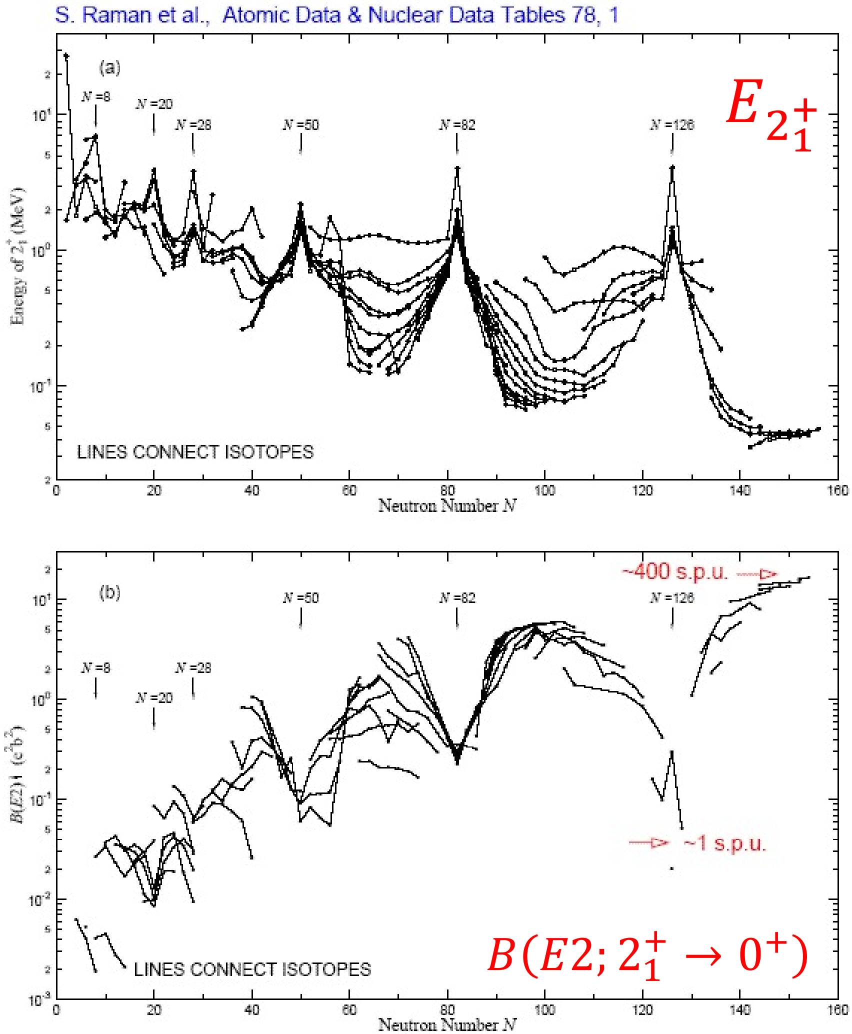


2-neutron binding energies = 2-neutron ´separation´energies

$$S_{2n} = BE(N,Z) - BE(N-2,Z)$$



Shell structure in nuclei



Nuclei with magic numbers of neutrons/protons

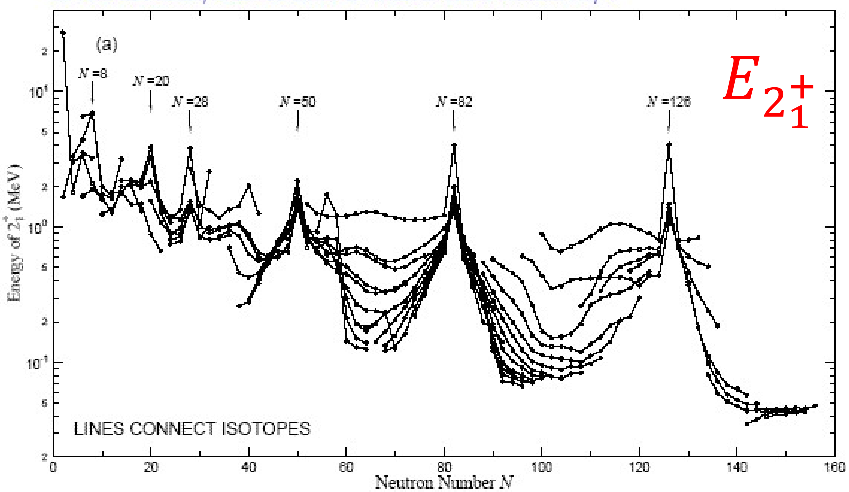
- ***high energies of the first excited 2^+ state***

- ***small nuclear deformations***

  transition probabilities measured in single particle units (spu)

Shell structure in nuclei

S. Raman et al., Atomic Data & Nuclear Data Tables 78, 1



Maria Goeppert-Mayer J. Hans D. Jensen

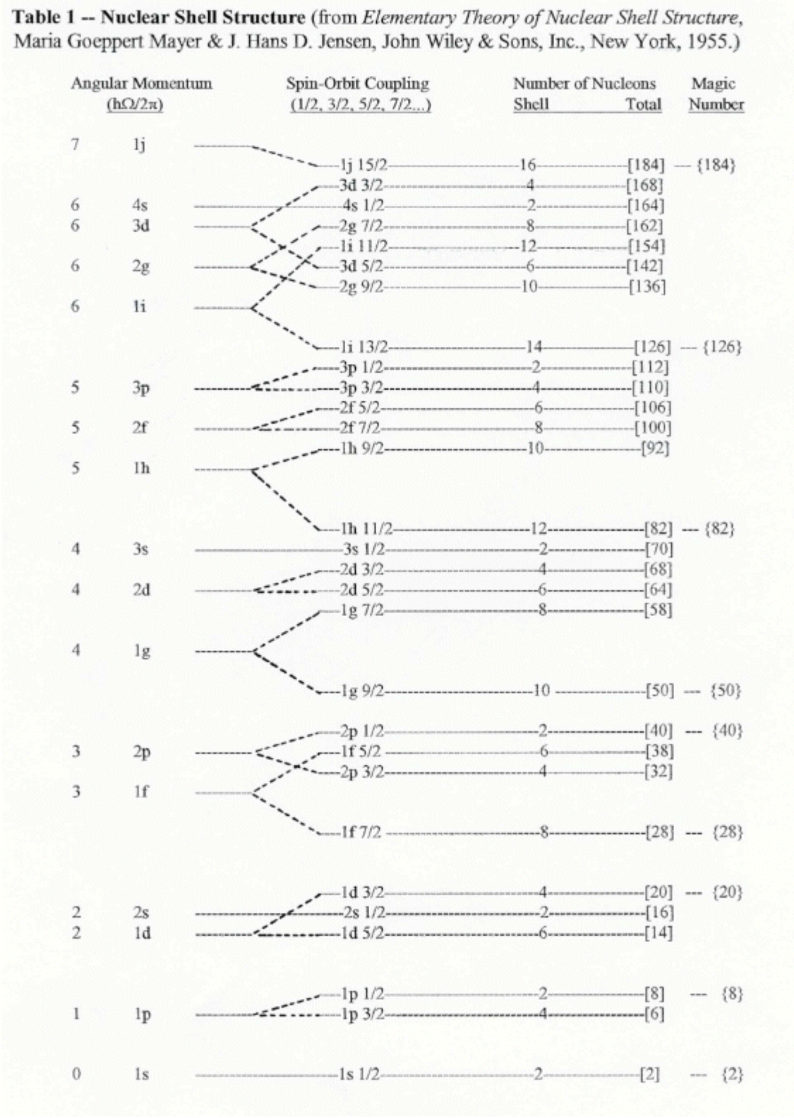


Table 1 -- Nuclear Shell Structure (from *Elementary Theory of Nuclear Shell Structure*, Maria Goeppert Mayer & J. Hans D. Jensen, John Wiley & Sons, Inc., New York, 1955.)

Angular Momentum (hΩ/2π)		Spin-Orbit Coupling (1/2, 3/2, 5/2, 7/2...)	Number of Nucleons Shell	Total	Magic Number
7	1j				
		1j 15/2	16	[184]	{184}
		3d 3/2	4	[168]	
6	4s	4s 1/2	2	[164]	
6	3d	2g 7/2	8	[162]	
		1i 11/2	12	[154]	
6	2g	3d 5/2	6	[142]	
		2g 9/2	10	[136]	
6	1i				
		1i 13/2	14	[126]	{126}
		3p 1/2	2	[112]	
5	3p	3p 3/2	4	[110]	
		2f 5/2	6	[106]	
5	2f	2f 7/2	8	[100]	
		1h 9/2	10	[92]	
5	1h				
		1h 11/2	12	[82]	{82}
4	3s	3s 1/2	2	[70]	
		2d 3/2	4	[68]	
4	2d	2d 5/2	6	[64]	
		1g 7/2	8	[58]	
4	1g				
		1g 9/2	10	[50]	{50}
		2p 1/2	2	[40]	{40}
3	2p	1f 5/2	6	[38]	
		2p 3/2	4	[32]	
3	1f				
		1f 7/2	8	[28]	{28}
		1d 3/2	4	[20]	{20}
2	2s	2s 1/2	2	[16]	
2	1d	1d 5/2	6	[14]	
		1p 1/2	2	[8]	{8}
1	1p	1p 3/2	4	[6]	
0	1s	1s 1/2	2	[2]	{2}

Nuclear potential

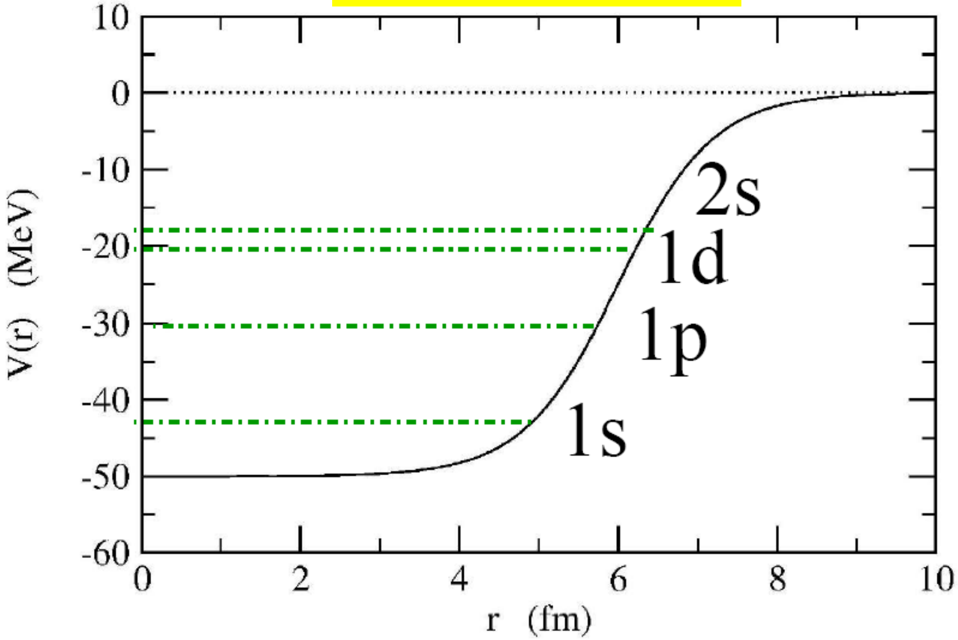
$$\hat{H} = \sum_{i=1}^{A} \frac{\hat{p}_i^2}{2m_i} + \sum_{i<j}^{A} \hat{V}(r_i, r_j)$$

$$\hat{H} = \sum_{i=1}^{A} \left[\frac{\hat{p}_i^2}{2m_i} + \hat{V}(r_i) \right] + \left[\sum_{i<j}^{A} \hat{V}(r_i, r_j) - \sum_{i=1}^{A} \hat{V}(r_i) \right]$$

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(r) - \varepsilon \right] \Psi(r) = 0$$

$$\Psi(r) = \frac{u_\ell(r)}{r} \cdot Y_{\ell m}(\vartheta, \varphi) \cdot X_{m_s}$$

$$V(r) = \frac{-V_0}{1 + e^{(r-R_0)/a}}$$



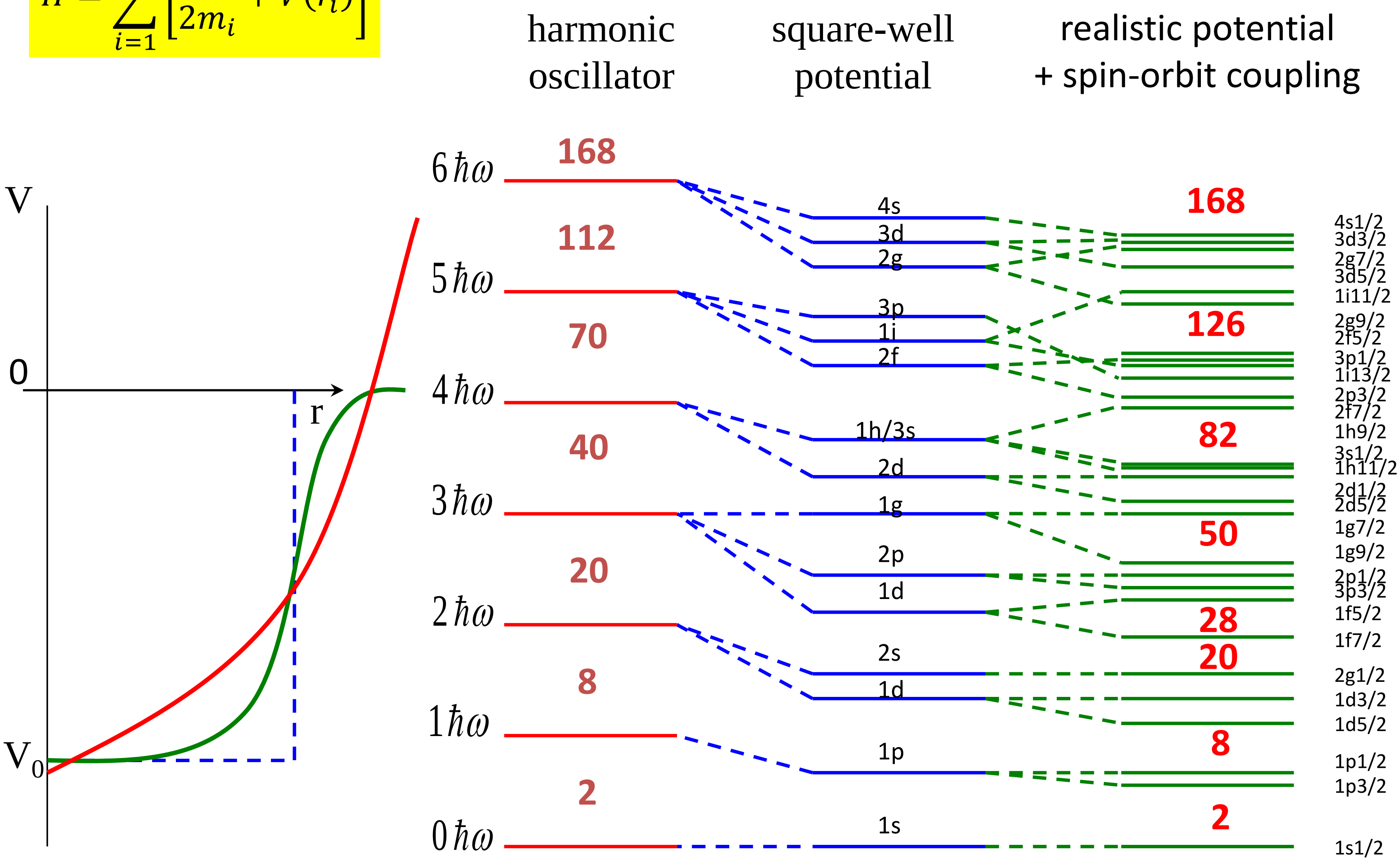
In the average nuclear potential V(r):

a) harmonic oscillator
b) square well potential
c) Woods-Saxon potential

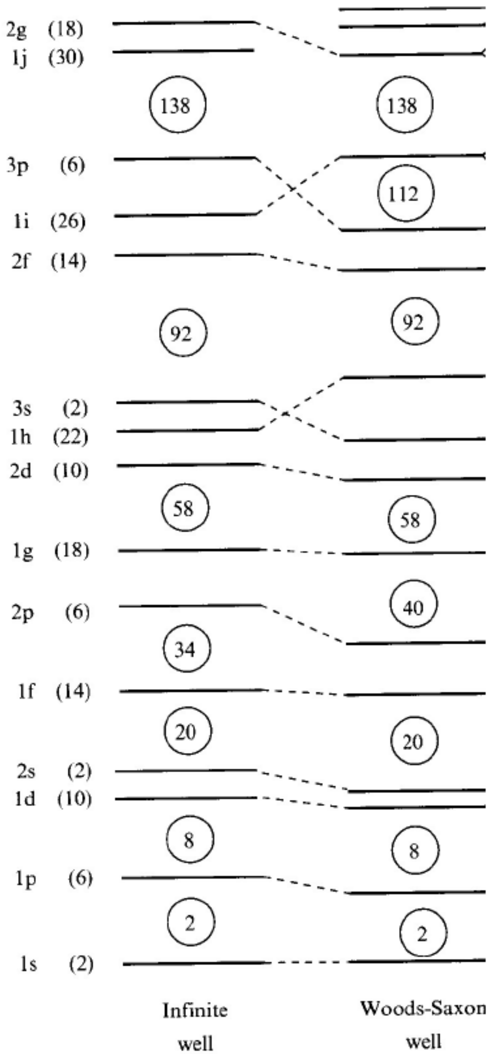
the nucleons move freely

Nuclear shell model

$$\hat{H} = \sum_{i=1}^{A} \left[\frac{\hat{p}_i^2}{2m_i} + \hat{V}(r_i) \right]$$



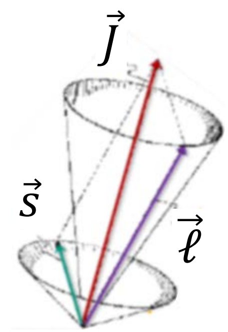
Woods-Saxon potential

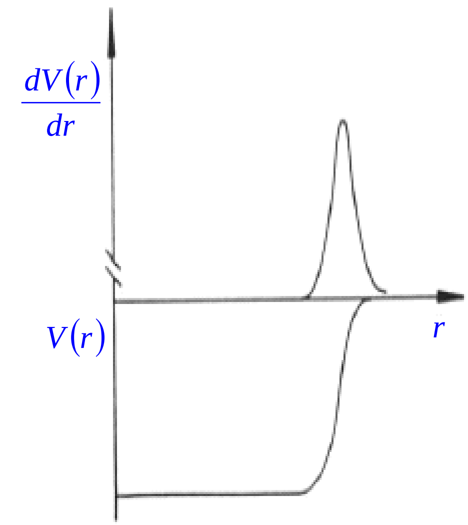


- ***Woods-Saxon does not reproduce the correct magic numbers***
  $(2, 8, 20, 40, 70, 112, 168)_{\mathrm{WS}}$ ***$(2, 8, 20, 28, 50, 82, 126)_{exp}$***
- ***Meyer und Jensen (1949): strong spin-orbit interaction***

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + V_{\ell s}(r)\cdot\vec{\ell}\cdot\vec{s} - \varepsilon\right]\Psi(r) = 0$$

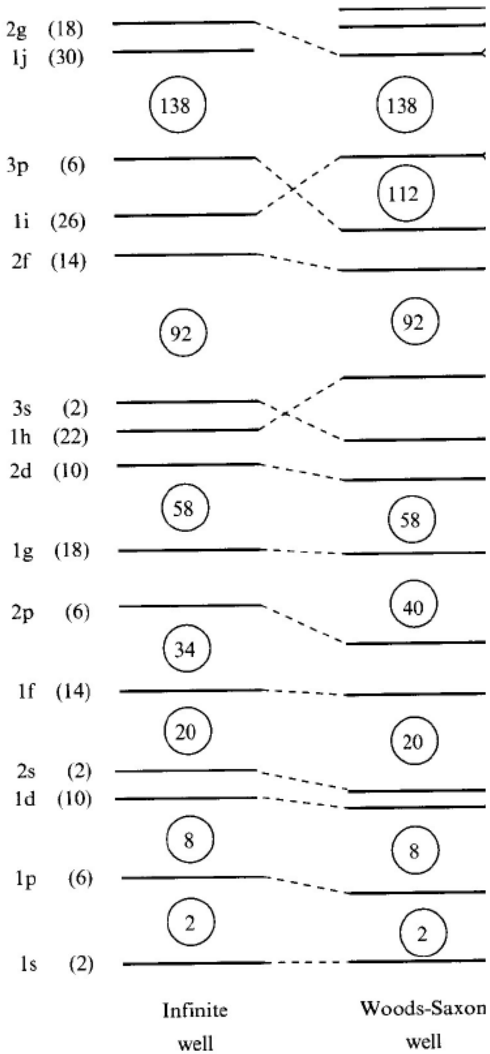
$$V_{\ell s}(r) \sim -\lambda\cdot\frac{1}{r}\cdot\frac{dV}{dr} \quad mit \quad \lambda > 0$$





The spin-orbit term has its origin in the relativistic description of the single particle motion inside the nucleus

Woods-Saxon potential (jj-coupling)



$\vec{j} = \vec{\ell} + \vec{s}$ $\Rightarrow$ $\langle \ell \cdot s \rangle = \frac{1}{2} \cdot [\langle j^2 \rangle - \langle \ell^2 \rangle - \langle s^2 \rangle] \cdot \hbar^2$

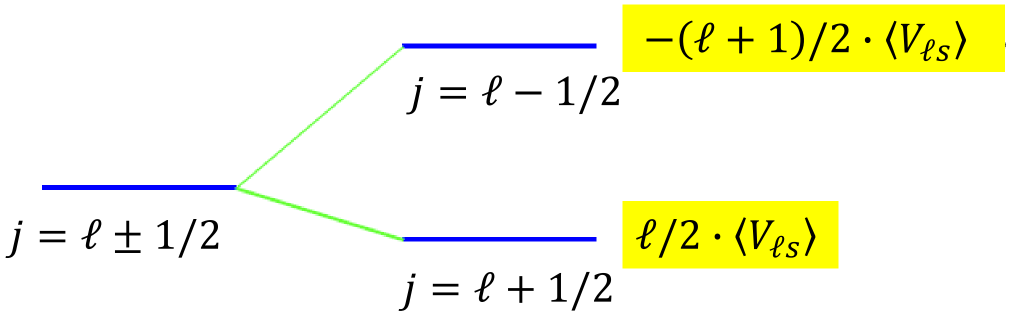
$$= \frac{1}{2}[j(j+1) - \ell(\ell+1) - s(s+1)] \cdot \hbar^2$$

The nuclear potential with spin-orbit term:

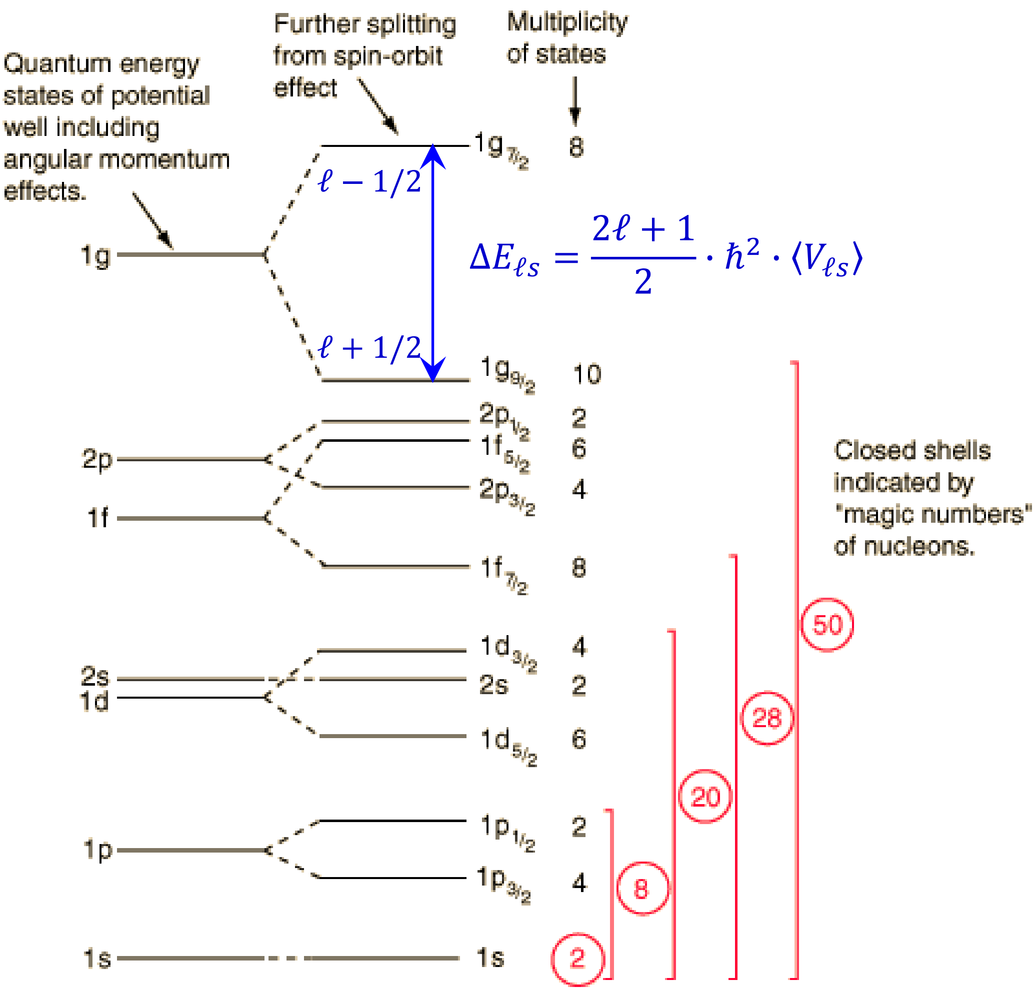
$$V(r) + \frac{\ell}{2} \cdot V_{\ell s} \quad for \quad j = \ell + 1/2$$

$$V(r) - \frac{\ell + 1}{2} V_{\ell s} \quad for \quad j = \ell - 1/2$$

spin-orbit interaction leads to a large splitting for large ℓ.



Woods-Saxon potential

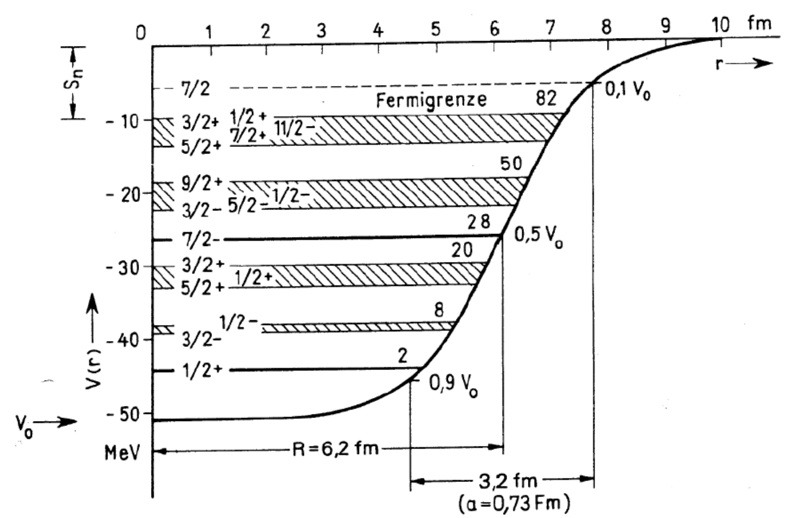


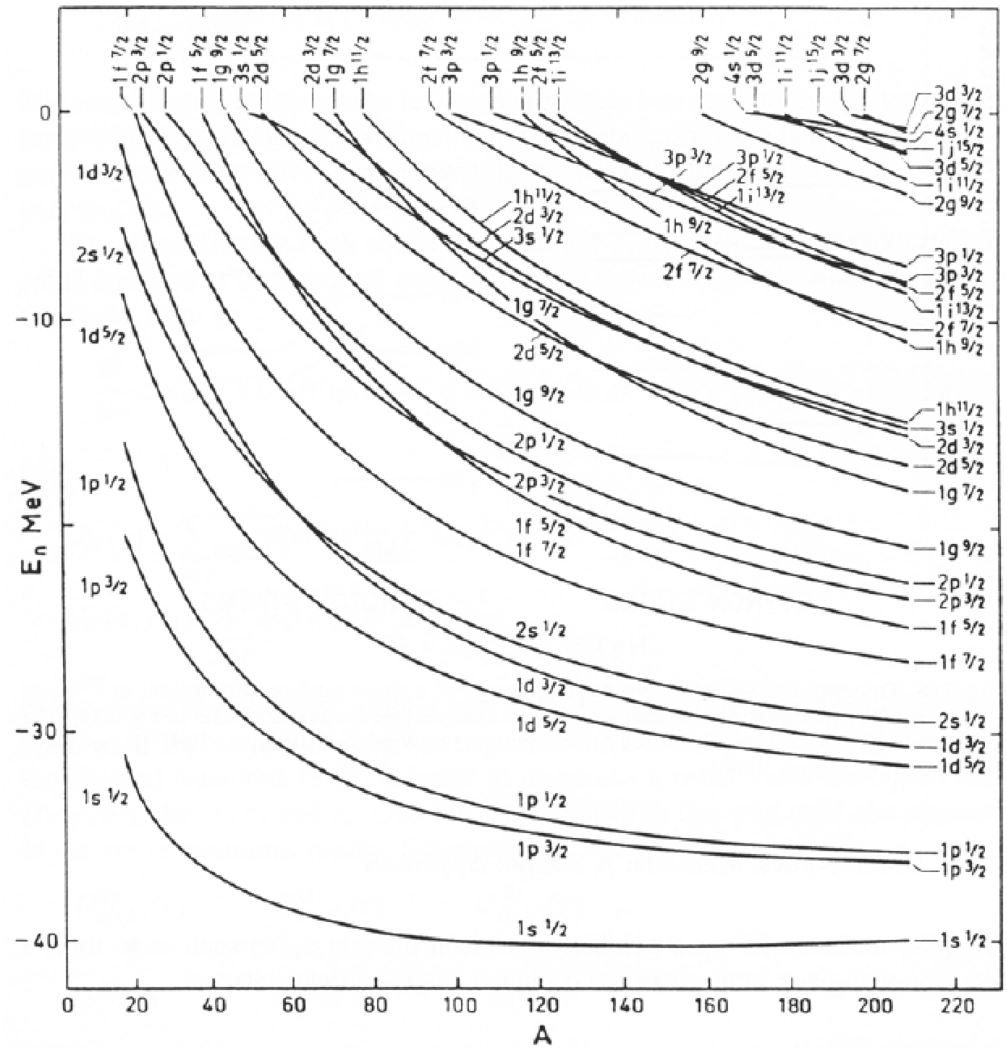
The spin-orbit term

- lowers the j = ℓ+1/2 orbital from the higher oscillator shell (intruder states)
- reproduces the magic numbers
  large energy gaps → very stable nuclei

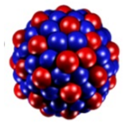
Important consequences:

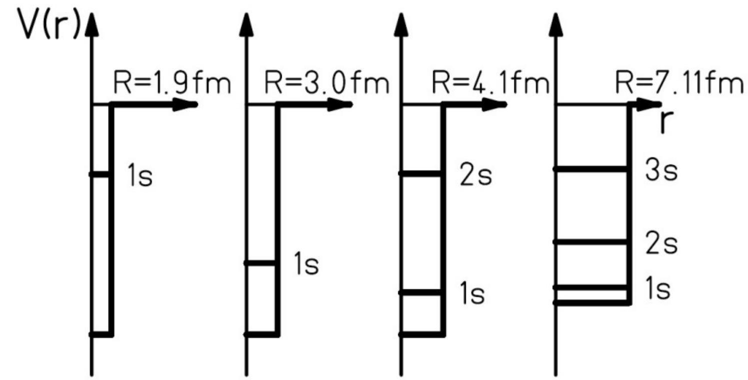
- lowering orbitals from higher lying N+1 shell having different parity than orbitals from the N shell
- strong interaction preserves the parity. The lowered orbitals with different parity are rather pure states and do not mix within the shell





- ***Mass dependence of the neutron energies:*** $E \sim R^{-2}$
- ***number of neutrons in each level:*** $2 \cdot (2\ell + 1)$





Success of the extreme single-particle shell model

Z	Isotope	Observed J^π	Shell model nlj
3	^{9}Li	$(3/2^-)$	$1p_{3/2}$
5	^{13}B	$3/2^-$	$1p_{3/2}$
7	^{17}N	$1/2^-$	$1p_{1/2}$
9	^{21}F	$5/2^+$	$1d_{5/2}$
11	^{25}Na	$5/2^+$	$1d_{5/2}$
13	^{29}Al	$5/2^+$	$1d_{5/2}$
15	^{33}P	$1/2^+$	$2s_{1/2}$
17	^{37}Cl	$3/2^+$	$1d_{3/2}$
19	^{41}K	$3/2^+$	$1d_{3/2}$
21	^{45}Sc	$7/2^-$	$1f_{7/2}$
23	^{49}Va	$7/2^-$	$1f_{7/2}$
25	^{53}Mn	$7/2^-$	$1f_{7/2}$
27	^{57}Co	$7/2^-$	$1f_{7/2}$
29	^{61}Cu	$3/2^-$	$2p_{3/2}$
31	^{65}Ga	$3/2^-$	$2p_{3/2}$
33	^{69}As	$(5/2^-)$	$1f_{5/2}$
35	^{73}Br	$(3/2^-)$	$1f_{5/2}$

- ***Ground state spin and parity:***

Every orbital has 2j+1 magnetic sub-states, completely filled orbitals have spin J=0, they do not contribute to the nuclear spin.

For a nucleus with one nucleon outside a completely occupied orbital the nuclear spin is given by the single nucleon.

$$n\ \ell\ \boldsymbol{j} \rightarrow \boldsymbol{J}$$

$$(-)^{\ell} = \pi$$

Success of the extreme single-particle shell model

${}^{7}_{4}Be$

${}^{17}_{9}F$

${}^{63}_{28}Ni$

${}^{61}_{29}Cu$

${}^{91}_{40}Zr$

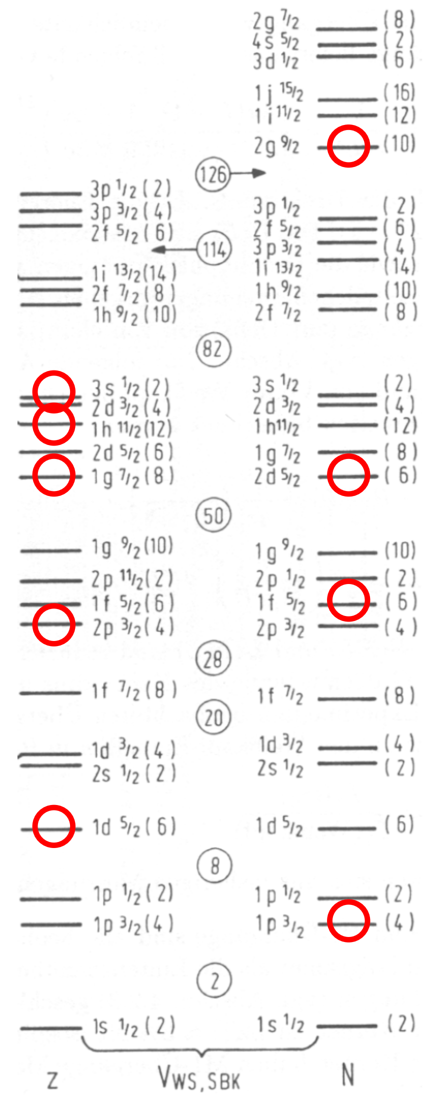
${}^{123}_{51}Sb$

${}^{159}_{65}Tb$

${}^{183}_{73}Ta$

${}^{199}_{81}Tl$

${}^{209}_{82}Pb$



Success of the extreme single-particle shell model

➢ ***Magnetic moments:***

The g-factor g_j is given by:

$$\overrightarrow{\mu_j} = g_\ell \cdot \vec{\ell} + g_s \cdot \vec{s} = g_j \cdot \vec{j}$$

$$\Rightarrow \quad \overrightarrow{\mu_j} = \left[\left(g_\ell \cdot \vec{\ell} + g_s \cdot \vec{s} \right) \cdot \frac{\vec{j}}{|j|} \right] \cdot \frac{\vec{j}}{|j|}$$

with $\vec{\ell}^2 = (\vec{j} - \vec{s})^2 = \vec{j}^2 - 2 \cdot \vec{j} \cdot \vec{s} + \vec{s}^2$ $\quad \vec{s}^2 = \left(\vec{j} - \vec{\ell}\right)^2 = \vec{j}^2 - 2 \cdot \vec{j} \cdot \vec{\ell} + \vec{\ell}^2$

$$\vec{\mu}_j = \frac{g_\ell \cdot \{j(j+1) + \ell(\ell+1) - 3/4\} + g_s \cdot \{j(j+1) - \ell(\ell+1) + 3/4\}}{2 \cdot j(j+1)} \cdot \vec{j}$$

$$g_j = \frac{1}{2} \cdot (g_\ell + g_s) + \frac{1}{2} \cdot \frac{\ell(\ell+1) - s(s+1)}{2j(j+1)} \cdot (g_\ell - g_s)$$

Simple relation for the g-factor of single-particle states

$$\frac{\mu}{\mu_N} = g_{nucleus} = g_\ell \pm \frac{(g_s - g_\ell)}{2\ell + 1} \quad for \quad j = \ell \pm 1$$

nucleus	state	J^π	μ/μ_N model	μ/μ_N experiment
^{15}N	p-$1p_{1/2}^{-1}$	$1/2^-$	$-0,264$	$-0,283$
^{15}O	n-$1p_{1/2}^{-1}$	$1/2^-$	$+0,638$	$+0,719$
^{17}O	n-$1d_{5/2}$	$5/2^+$	$-1,913$	$-1,894$
^{17}F	p-$1d_{5/2}$	$5/2^+$	$+4,722$	$+4,793$

Success of the extreme single-particle shell model

➢ ***magnetic moments:***

$$\langle \mu_z \rangle = \begin{cases} \left[g_\ell \cdot \left(j - \frac{1}{2} \right) + \frac{1}{2} \cdot g_s \right] \cdot \mu_N & for \quad j = \ell + 1/2 \\ \frac{j}{j+1} \cdot \left[g_\ell \cdot \left(j + \frac{3}{2} \right) - \frac{1}{2} \cdot g_s \right] \cdot \mu_N & for \quad j = \ell - 1/2 \end{cases}$$

➢ ***g-factor of nukleons:***

proton: $g_\ell = 1$; $g_s = +5.585$

neutron: $g_\ell = 0$; $g_s = -3.82$

proton:

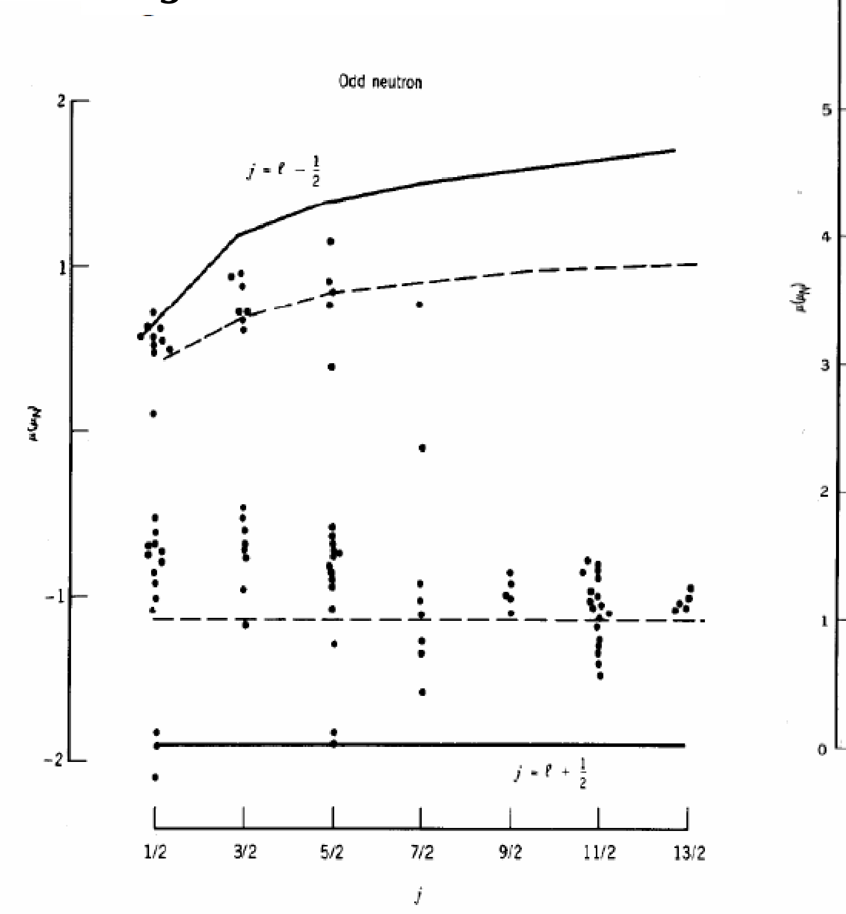
$$\langle \mu_z \rangle = \begin{cases} (j + 2.293) \cdot \mu_N & for \quad j = \ell + 1/2 \\ (j - 2.293) \cdot \frac{j}{j+1} \cdot \mu_N & for \quad j = \ell - 1/2 \end{cases}$$

neutron:

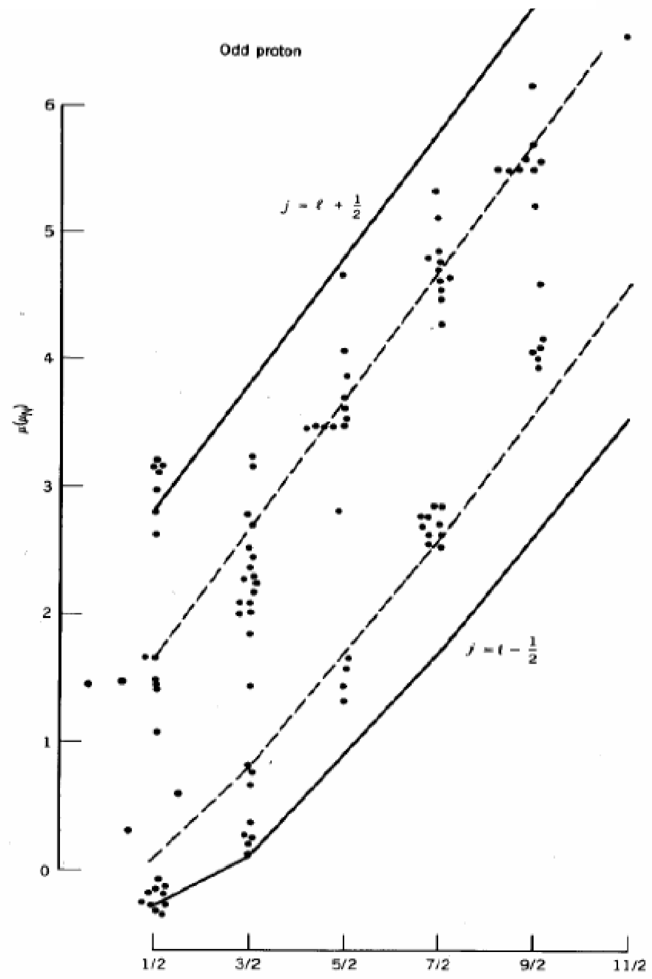
$$\langle \mu_z \rangle = \begin{cases} -1.91 \cdot \mu_N & for \quad j = \ell + 1/2 \\ +1.91 \cdot \frac{j}{j+1} \cdot \mu_N & for \quad j = \ell - 1/2 \end{cases}$$

Magnetic moments: Schmidt lines

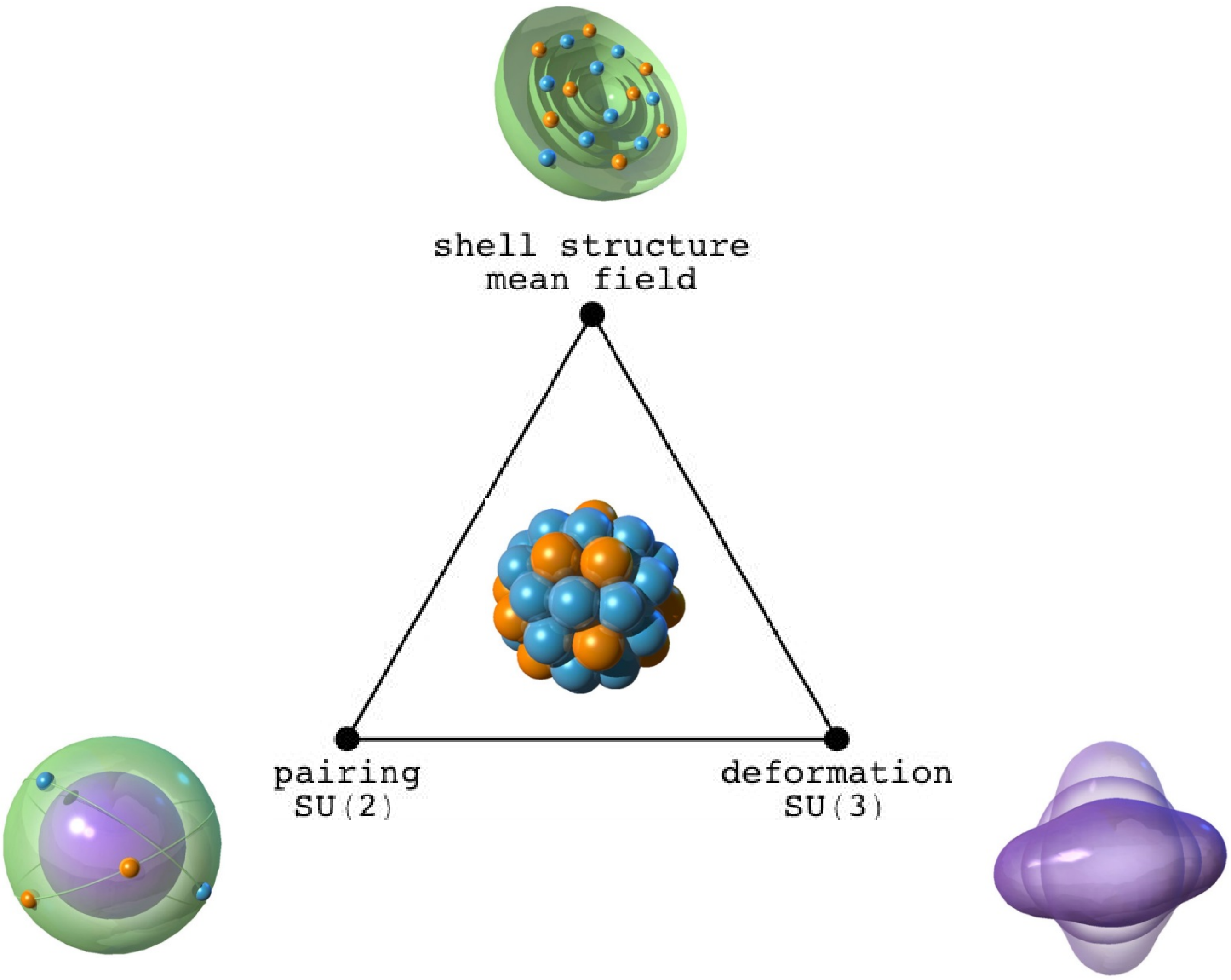
magnetic moments: neutron



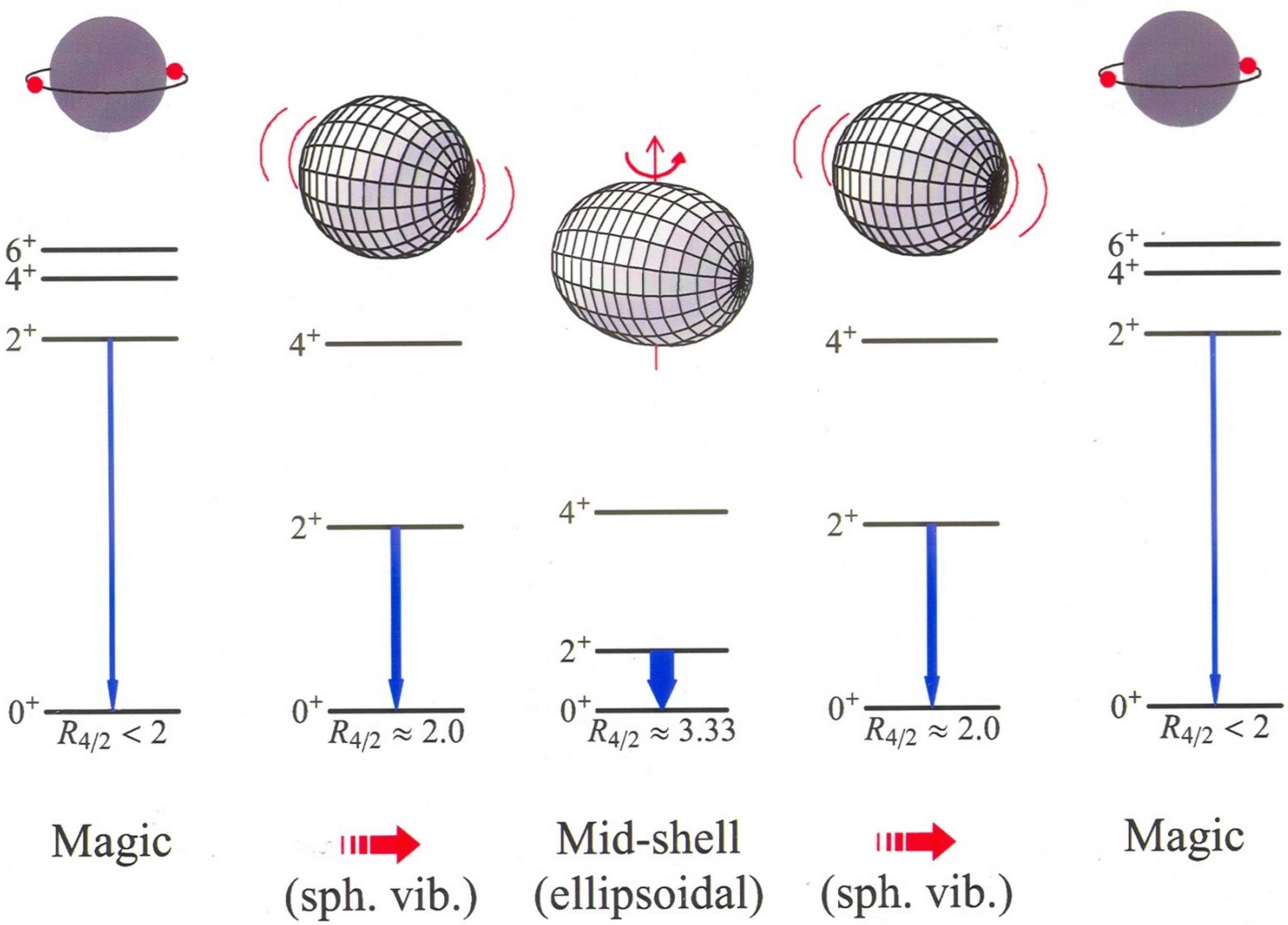
magnetic moments: proton



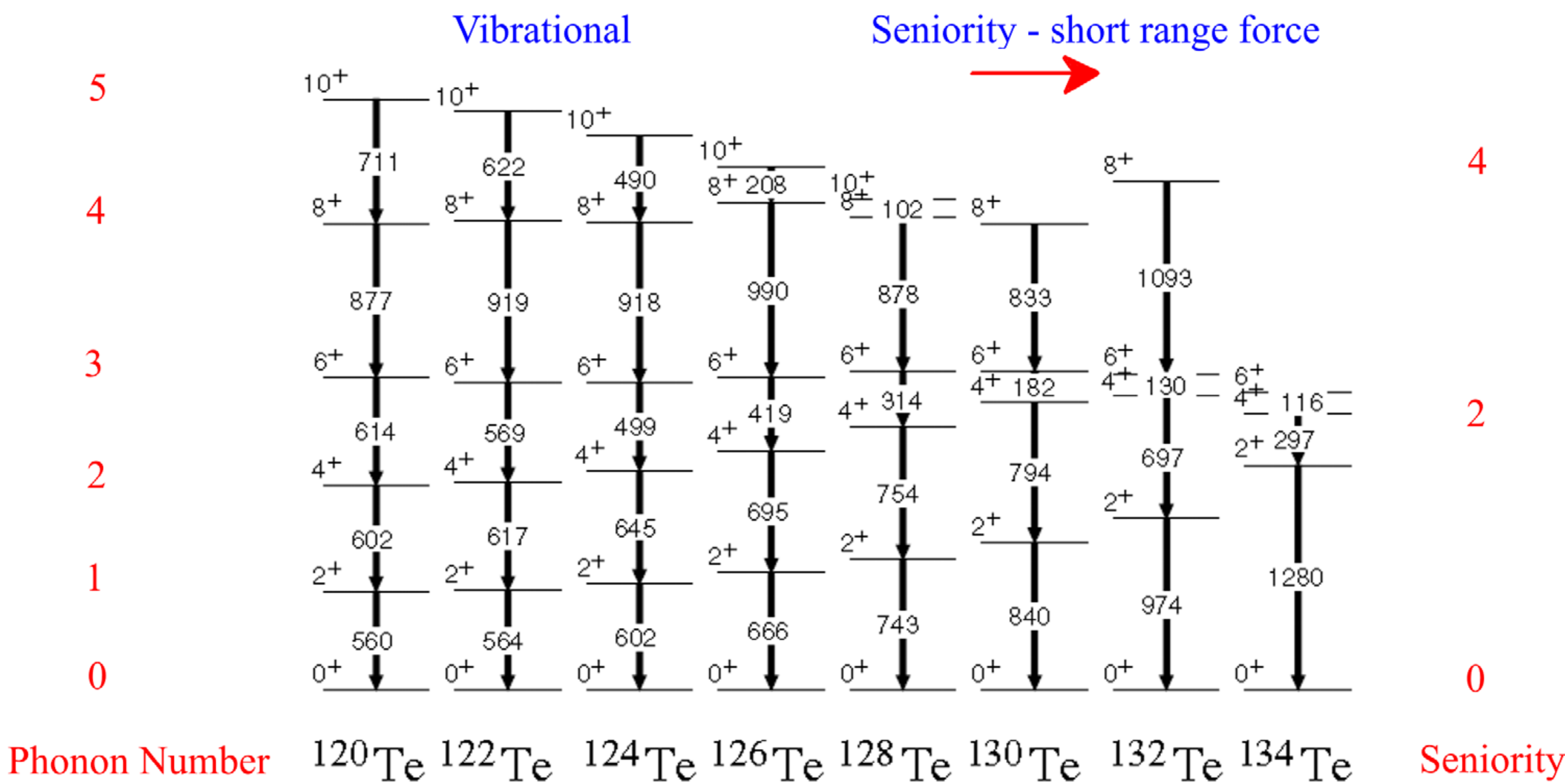
The three structures of the shell model



Evolution of nuclear structure (as a function of nucleon number)



Systematics of the Te isotopes (Z=52)



Neutron number	68	70	72	74	76	78	80	82
Val. Neutr. number	14	12	10	8	6	4	2	0