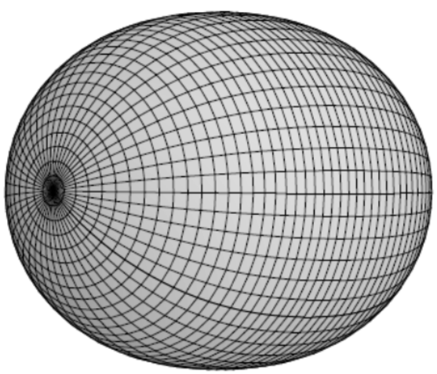
Shape parameterization

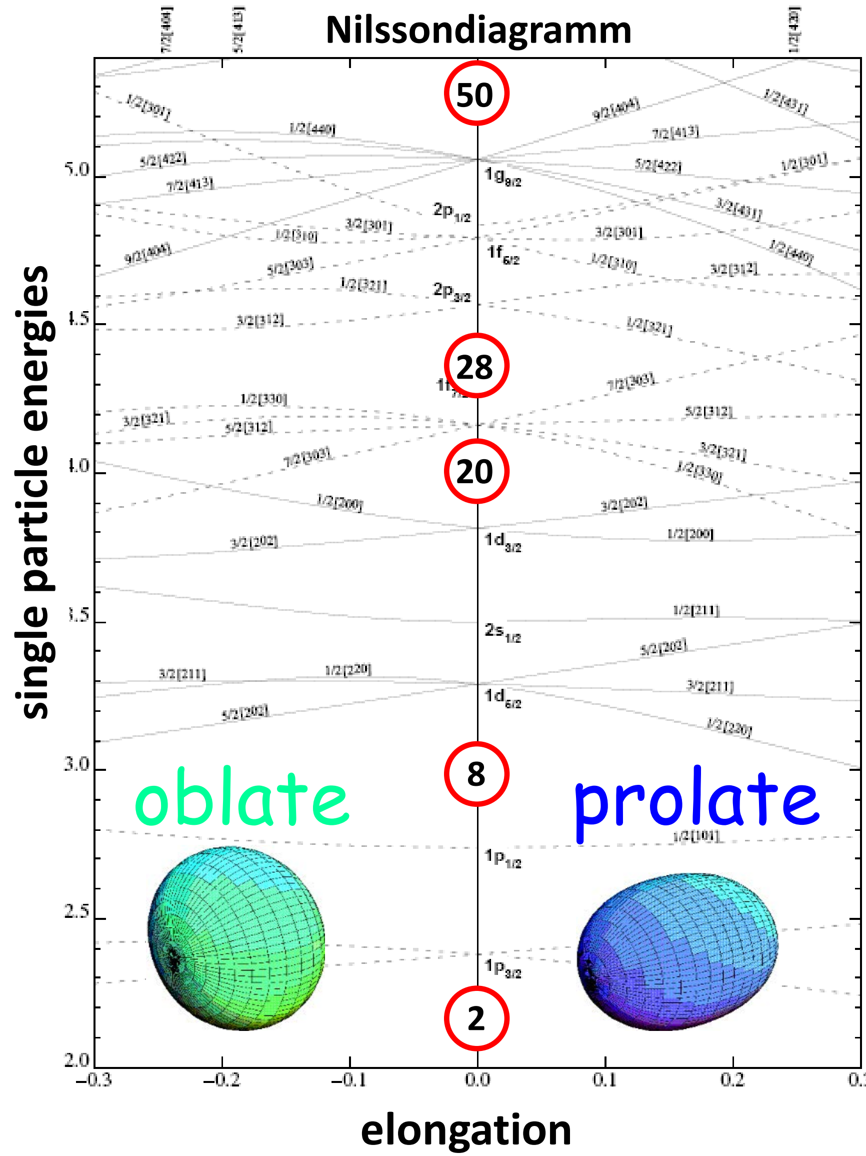
$$R(\theta,\phi) = R_0 \cdot \left[1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu} \cdot Y_{\lambda\mu}(\theta,\phi) \right]$$

axially symmetric **quadrupole**



λ=2

$\alpha_{20} \neq 0,\ \alpha_{2\pm1} = \alpha_{2\pm2} = 0$



Quadrupole deformation (λ=2)

$$R(\theta,\phi) = R_0 \cdot \left[1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu} \cdot Y_{\lambda\mu}(\theta,\phi) \right]$$

There are five independent real parameters,

- α_{20} indicates the stretching of the 3-axis with respect to the 1- and 2-axes
- α_{22} determines the difference in length between the 1- and 2-axes
- three Euler angles, which determine the orientation of the principle axis system (1,2,3) with respect to the laboratory frame (x,y,z)

Hill - Wheeler introduced the (β, γ) – parameters:

$$a_{20} = \beta_2 \cos\gamma$$

$$a_{22} = \frac{1}{\sqrt{2}} \beta_2 \sin\gamma$$

Quadrupole deformation (λ=2)

$$R(\theta,\phi) = R_0 \cdot \left\{ 1 + \beta \cdot \cos\gamma \cdot Y_{20}(\theta,\phi) + \frac{1}{\sqrt{2}} \cdot \beta \cdot \sin\gamma \cdot \left[Y_{22}(\theta,\phi) + Y_{2-2}(\theta,\phi) \right] \right\}$$

$$R(\theta,\phi) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{16\pi}} \left[\cos\gamma \cdot \left(3 \cdot \cos^2\theta - 1 \right) + \sqrt{3} \cdot \sin\gamma \cdot \sin^2\theta \cdot \cos 2\phi \right] \right\}$$

Consider the nuclear shapes in the principal axis system (1, 2, 3) ≡ (x´, y´, z´)

$$R_1 = R_{x'} = R\left(\frac{\pi}{2}, 0 \right) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{16\pi}} \left[-\cos\gamma + \sqrt{3} \cdot \sin\gamma \right] \right\}$$

$$R_2 = R_{y'} = R\left(\frac{\pi}{2}, \frac{\pi}{2} \right) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{16\pi}} \left[-\cos\gamma - \sqrt{3} \cdot \sin\gamma \right] \right\}$$

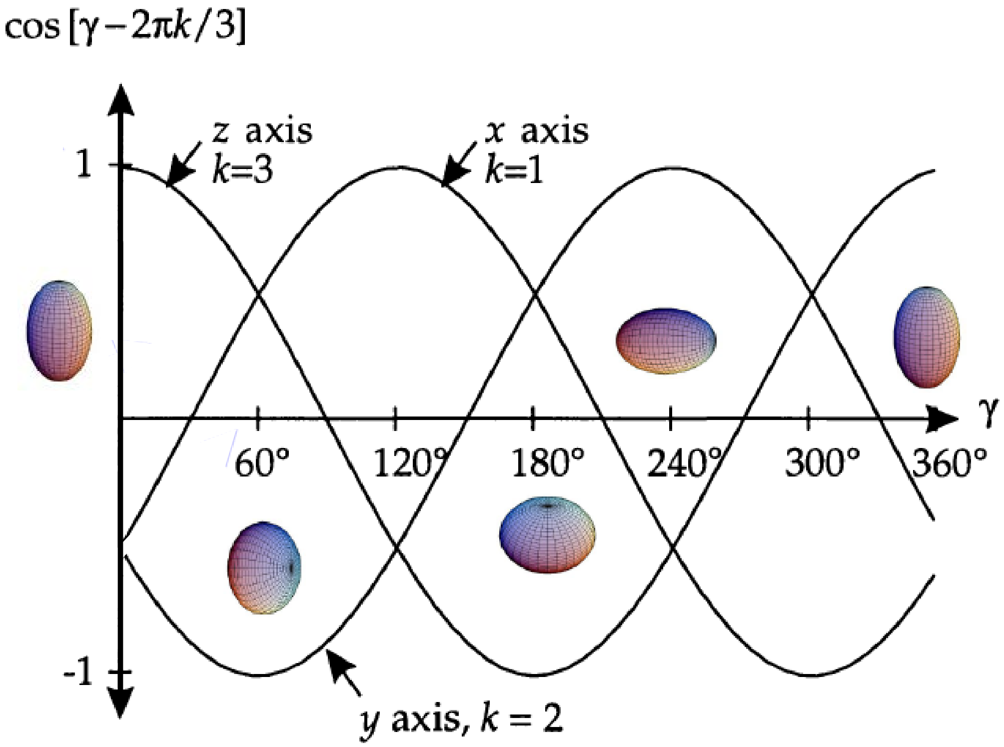
$$R_3 = R_{z'} = R(0,0) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{16\pi}} \cdot 2 \cdot \cos\gamma \right\}$$

$$R_k(\theta,\phi) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{4\pi}} \cdot \cos\left(\gamma - \frac{2\pi \cdot k}{3} \right) \right\} \quad for \;\; k = 1, 2, 3$$

(β,γ) coordinates

$$R_k(\theta,\phi) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{4\pi}} \cdot \cos\left(\gamma - \frac{2\pi \cdot k}{3} \right) \right\} \quad for \;\; k = 1,2,3$$

- At $\gamma = 0^0$ the nucleus is elongated along the z´axis, but the x´and y´axes are equal (**prolate shape** for x´= y´)
- As we increase γ, the x´axis grows at the expense of the y´and z´axes through a region of **triaxial shapes** with three unequal axis, until axial symmetry is again reached at $\gamma = 60^0$, but now with the z´and x´axis equal in length. These two axes are longer than the y´axis (**oblate shape** for x´= z´)
- This pattern is **repeated: every 60^0** axial symmetry repeated and prolate and oblate shapes alternate.



(β,γ) coordinates

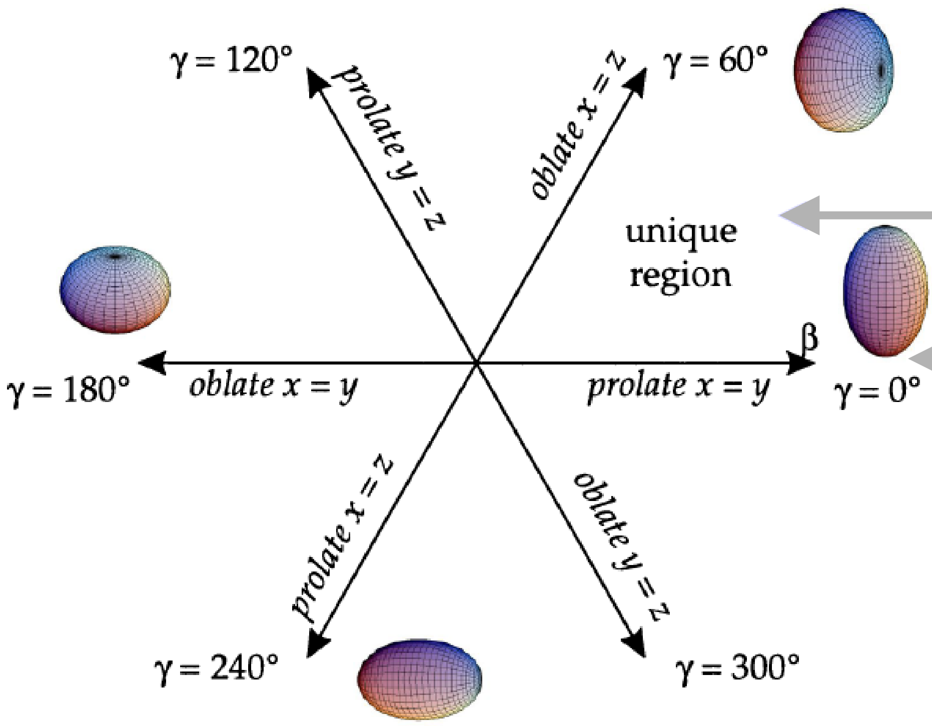


Figure: The (β,γ) plane is divided into six equivalent parts by the symmetries:

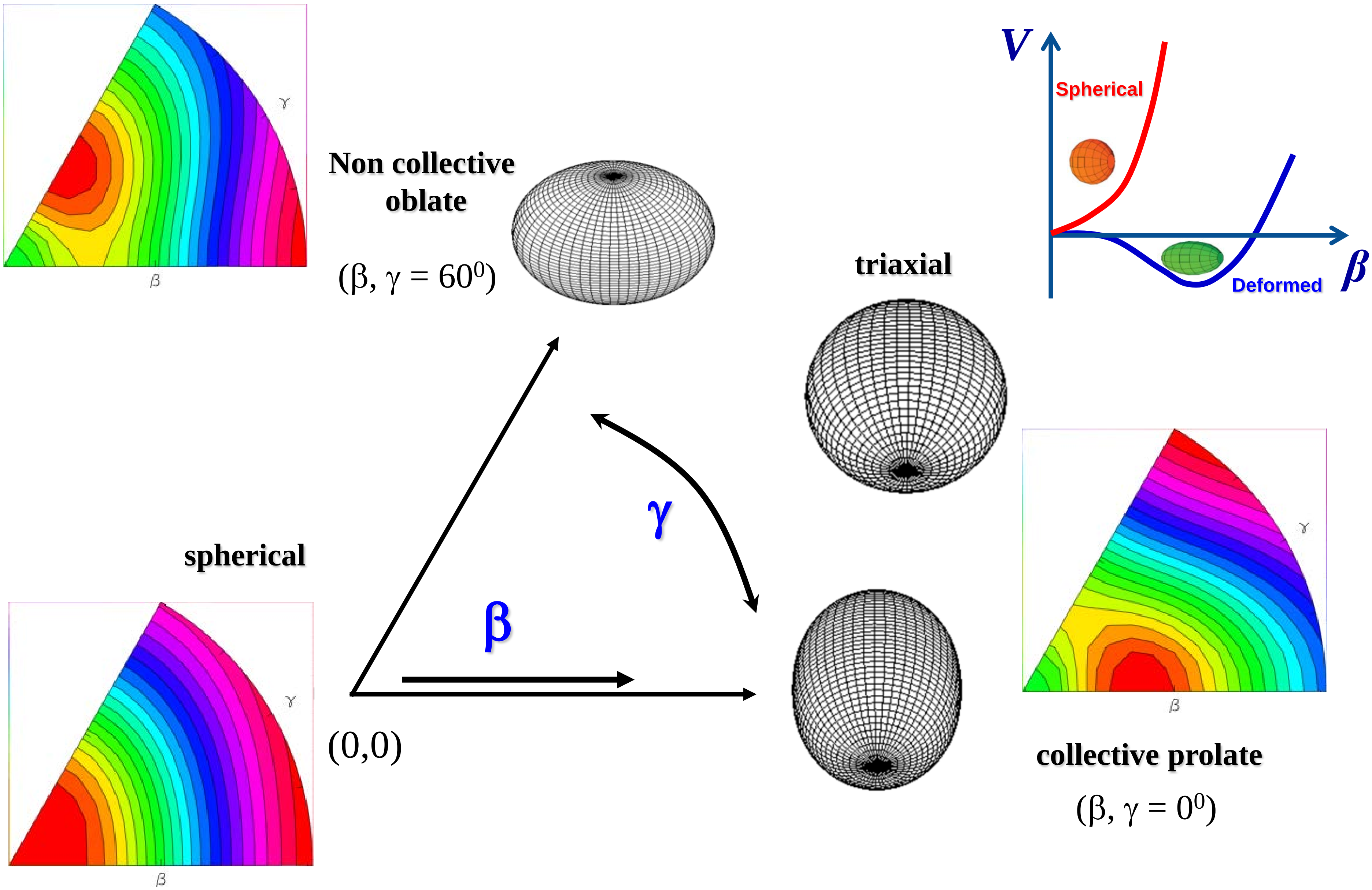
the sector 0^0 and 60^0 contains all shapes uniquely, i.e. **triaxial shapes**

the types of shapes encountered along the axis: e.g. **prolate x´= y´** implies prolate shapes with the z´axis as the long axis and the two other axis x´and y´equal.

⇨ various nuclear shapes – **prolate or oblate** – in the (β,γ) plane **are repeated every 60^0**. Because the axis orientations are different, the associated Euler angles also differ.

In conclusion, the same physical shape (including ist orientation in space) can be represented by different sets of deformation parameters (β,γ) and Euler angles!

(β,γ) coordinates



Collective excitation

E(4⁺) / E(2⁺): rotational vs vibrational

- **Rotational (deformed):**

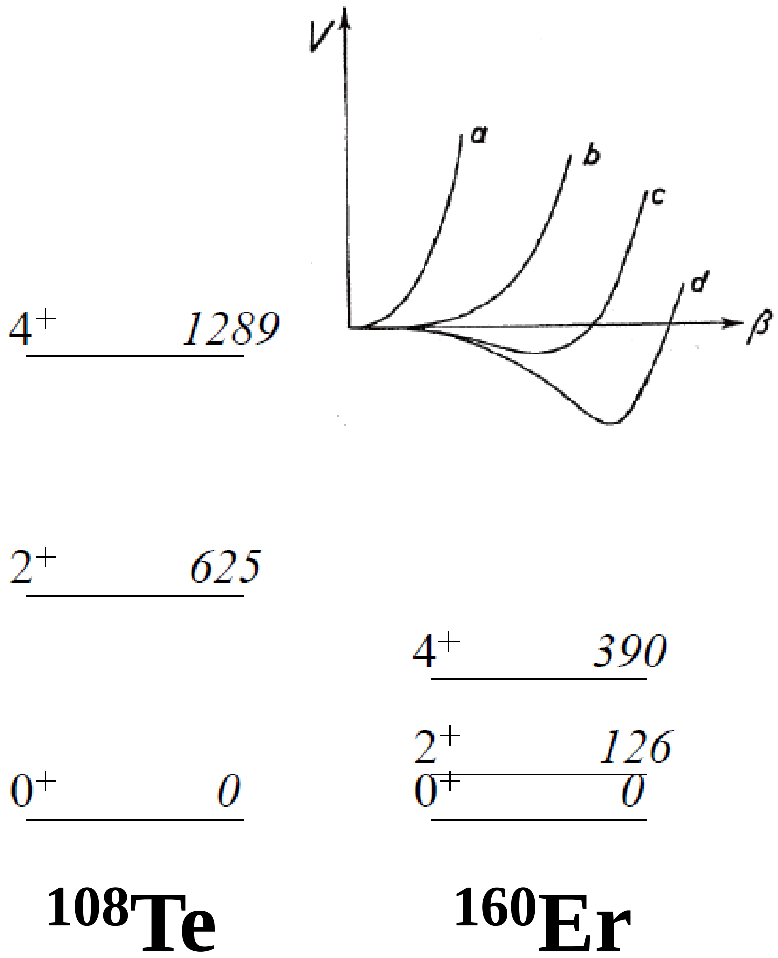
$$E_x(I) = \frac{I(I+1)\hbar^2}{2\cdot\mathfrak{I}}$$

  – $E(4^+)/E(2^+) = 10/3$

- **Vibrational (spherical):**

$$E_n = n\cdot\hbar\omega_2$$

  – $E(4^+)/E(2^+) = 2$



$$\frac{E(4^+)}{E(2^+)} = \quad \mathbf{2.1} \qquad \mathbf{3.1}$$

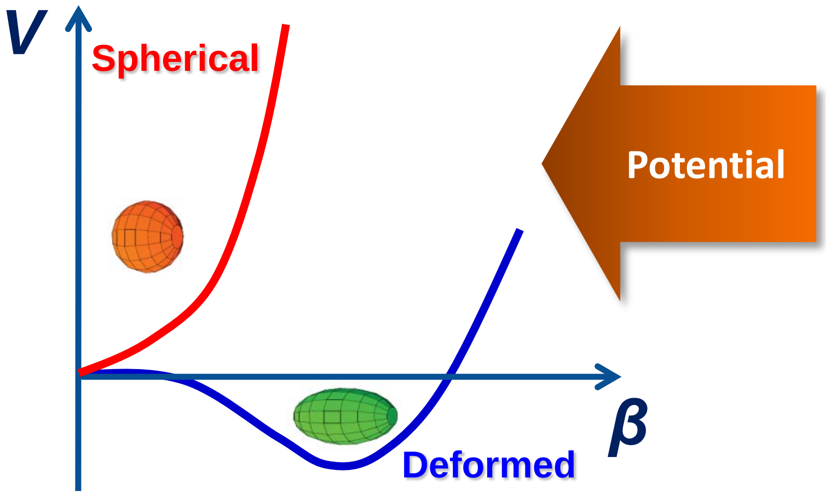
Classical collective Hamiltonian of Bohr-Mottelson for quadrupole deformation

$$H_{coll} = T_{vib} + T_{rot} + V_{coll} = \frac{1}{2}\sum_{\lambda\mu} B_\lambda \left|\dot{\alpha}_{\lambda\mu}\right|^2 + \frac{1}{2}\sum_{k=1}^{3} \Im_k \cdot \omega_k^2 + \frac{1}{2}\sum_{\lambda\mu} C_\lambda \left|\alpha_{\lambda\mu}\right|^2$$

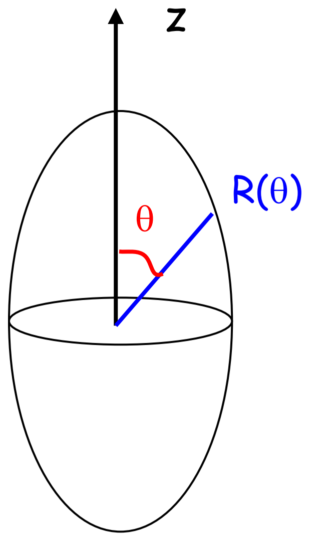
Quadrupole (λ=2) motion

$$H = \frac{1}{2} B \cdot (\dot{\beta}^2 + \beta^2 \cdot \dot{\gamma}^2) + \frac{1}{2}\sum_k \Im_k \cdot \omega_k^2 + \frac{1}{2} C \cdot \beta^2$$

where (β, γ) parameters have been used.

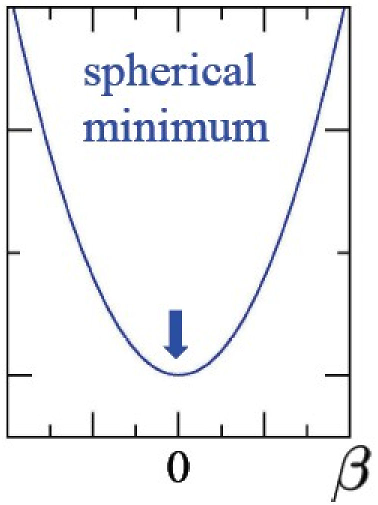


Moment of inertia



$$R(\theta) = R_0 \cdot [1 + \beta \cdot Y_{20}(\theta)]$$

$$\beta = \frac{4}{3} \cdot \sqrt{\frac{\pi}{5}} \cdot \frac{R(0^0) - R(90^0)}{R_0} \cong 1.05 \cdot \frac{\Delta R}{R_0}$$



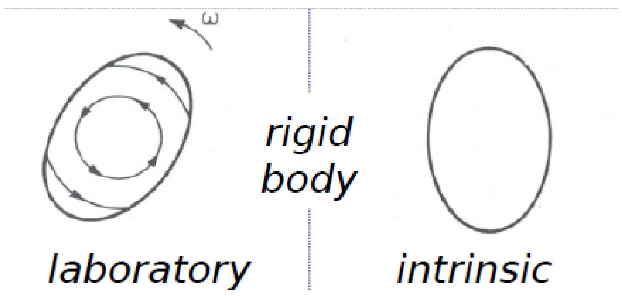
Rigid body moment of inertia:

$$\Im_R = \iiint r^2 \cdot \rho(r) \cdot r^2 dr \sin\theta \, d\theta \, d\phi$$

$$\Im_R = \frac{2}{5} M \; R_o^2 \; (1 + 0.32\beta)$$

Irrotational flow moment of inertia:

$$\Im_F = \frac{9}{8\pi} \; M \; R_o^2 \; \beta^2$$

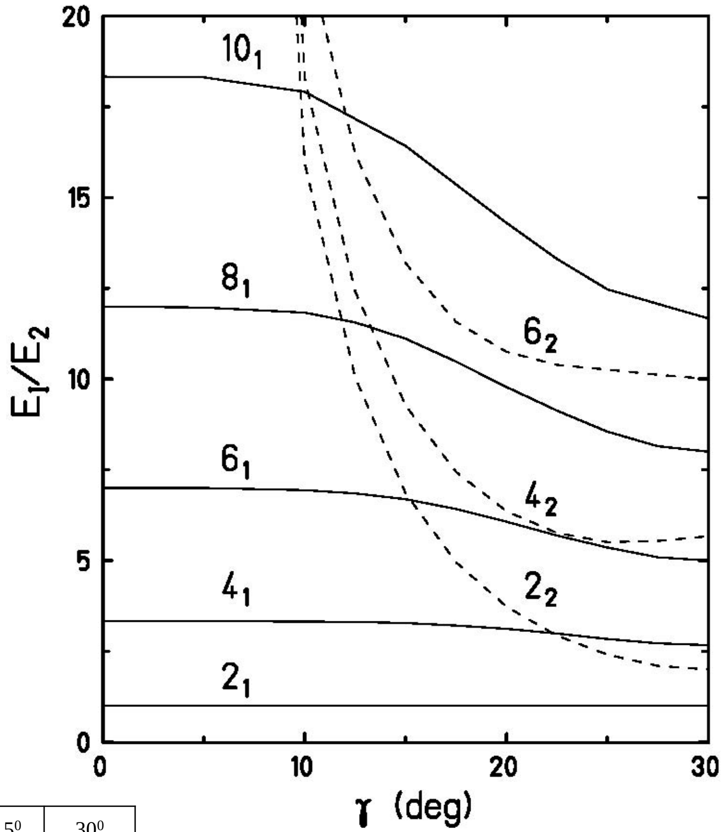


Experimental 2_2^+ energy and estimate of γ-deformation parameter

rigid triaxial rotor model

$$\frac{E(2_2)}{E(2_1)} = \frac{3+\sqrt{9-8\cdot\sin^2(3\gamma)}}{3-\sqrt{9-8\cdot\sin^2(3\gamma)}} \geq 2$$

Davydov and Filippov, Nucl. Phys. 8, 237 (1958)



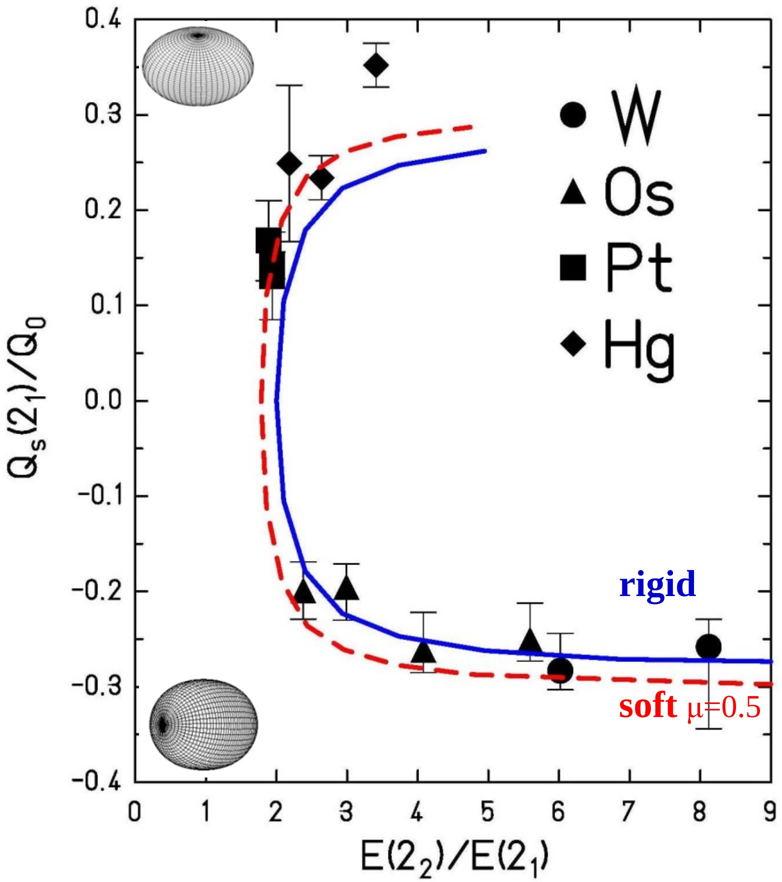
γ	0^0	5^0	10^0	12.5^0	15^0	17.5^0	20^0	22.5^0	25^0	27.5^0	30^0
$E(4_1)/E(2_1)$	3.33	3.33	3.32	3.31	3.28	3.21	3.12	2.99	2.84	2.72	2.67
$E(6_1)/E(2_1)$	7.00	7.00	6.94	6.85	6.69	6.42	6.07	5.69	5.36	5.09	5.00
$E(8_1)/E(2_1)$	12.00	11.97	11.83	11.56	11.11	10.48	9.78	9.13	8.55	8.15	8.00
$E(10_1)/E(2_1)$	18.33	18.31	17.91		16.42		14.30	13.31	12.47		11.67
$E(2_2)/E(2_1)$	∞	65.16	15.94	10.04	6.85	4.95	3.73	2.93	2.41	2.10	2.00
$E(4_2)/E(2_1)$	∞	67.50	18.28	12.41	9.27	7.44	6.36	5.76	5.51	5.54	5.67
$E(6_2)/E(2_1)$	∞	71.17	22.00	16.20	13.19	11.57	10.75	10.39	10.26	10.12	10.00

rigid triaxial rotor model

$$\frac{Q_s(2_1)}{Q_0} = -\frac{6\cdot\cos(3\gamma)}{7\cdot\sqrt{9-8\cdot\sin^2(3\gamma)}}$$

Davydov and Filippov, Nucl. Phys. 8, 237 (1958)

$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I\cdot(2I-1)}{(I+1)\cdot(2I+1)\cdot(2I+3)}}\cdot\frac{\langle I\|M(E2)\|I\rangle}{\langle 2_1\|M(E2)\|0_1\rangle}$$



γ	0^0	10^0	15^0	20^0	22.5^0	25^0	27.5^0	30^0
$Q_s(2_1)/Q_0$	-0.28	-0.28	-0.27	-0.25	-0.22	-0.18	-0.10	0.0

soft asymmetric rotor model: $\gamma \rightarrow \gamma_{eff} = \sqrt{\Gamma^2 + \gamma_o^2}$ with $\Gamma = \left\{\langle 0|(\gamma-\gamma_0)^2|0\rangle\right\}^{1/2}$ $\mu = \left\{\frac{\langle 0|(\beta-\beta_0)^2|0\rangle}{\beta_0^2}\right\}^{1/2}$

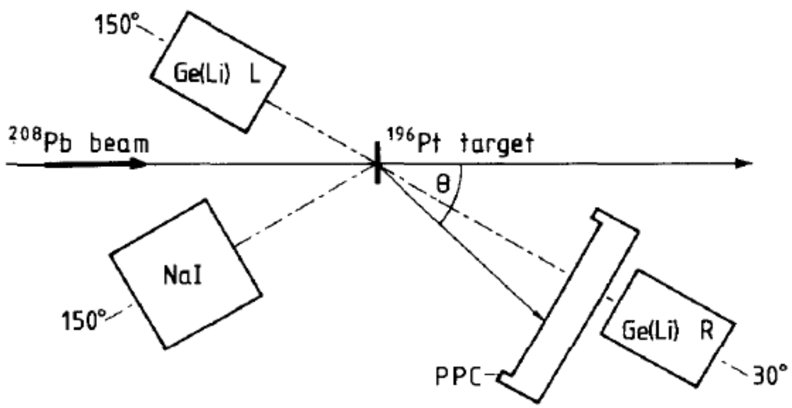
Experimental 2_2^+ B(E2)-values and estimate of γ-deformation parameter

rigid triaxial rotor model

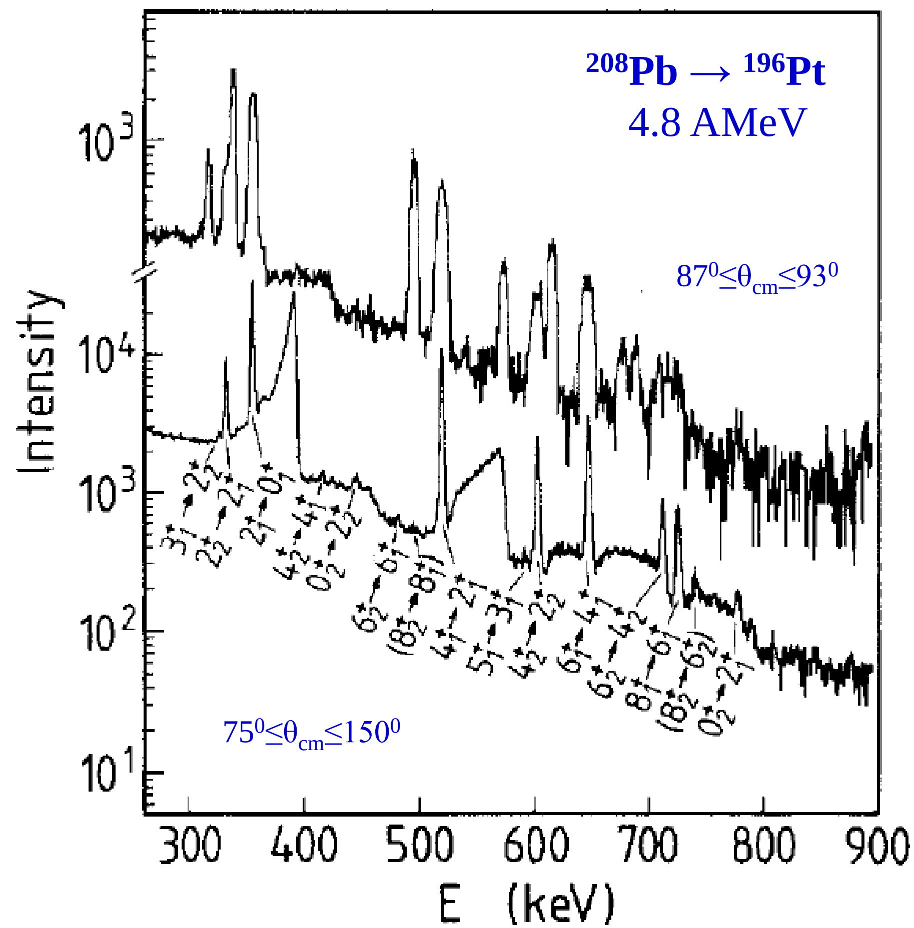
$$\frac{B(E2;2_2 \to 0)}{B(E2;2_1 \to 0)} = \frac{1 - \dfrac{3-2\sin^2(3\gamma)}{\sqrt{9-8\sin^2(3\gamma)}}}{1 + \dfrac{3-2\sin^2(3\gamma)}{\sqrt{9-8\sin^2(3\gamma)}}}$$

$$\frac{B(E2;2_2 \to 2_1)}{B(E2;2_1 \to 0)} = \frac{\dfrac{20}{7}\,\dfrac{\sin^2(3\gamma)}{9-8\sin^2(3\gamma)}}{1 + \dfrac{3-2\sin^2(3\gamma)}{\sqrt{9-8\sin^2(3\gamma)}}}$$

γ	0^0	5^0	10^0	15^0	20^0	25^0	30^0
$B(E2;2_2 \to 0)/B(E2;2_1 \to 0)$	0	.0075	.0288	.0560	.0718	.0445	0
$B(E2;2_2 \to 0)/B(E2;2_1 \to 0)$	0	.0111	.0525	.1510	.3826	.9058	1.43
$B(E2;2_2 \to 2)/B(E2;2_2 \to 0)$	1.43	1.49	1.70	2.70	5.35	20.6	∞

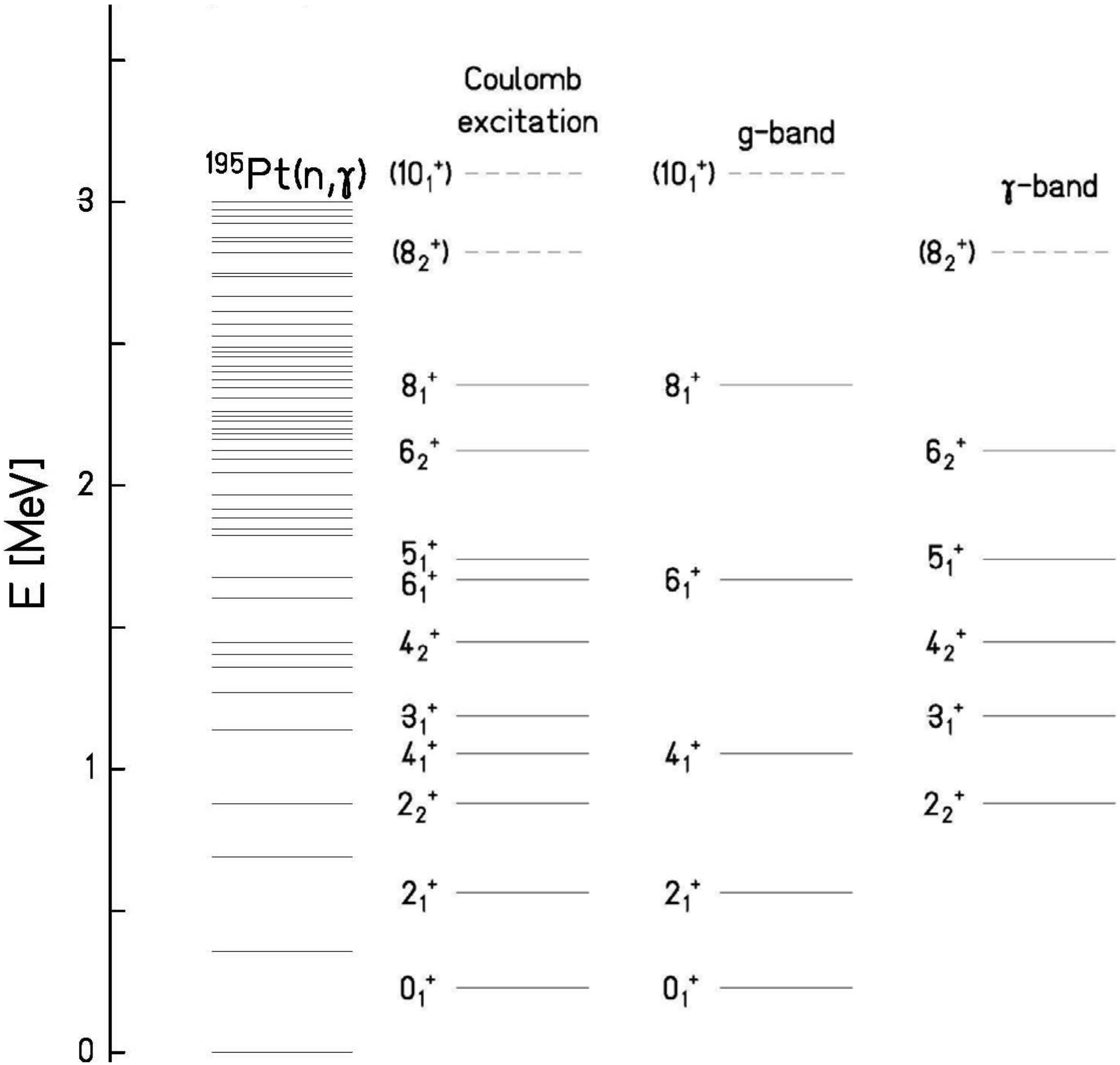


position-sensitive
parallel plate counter
(Doppler correction)

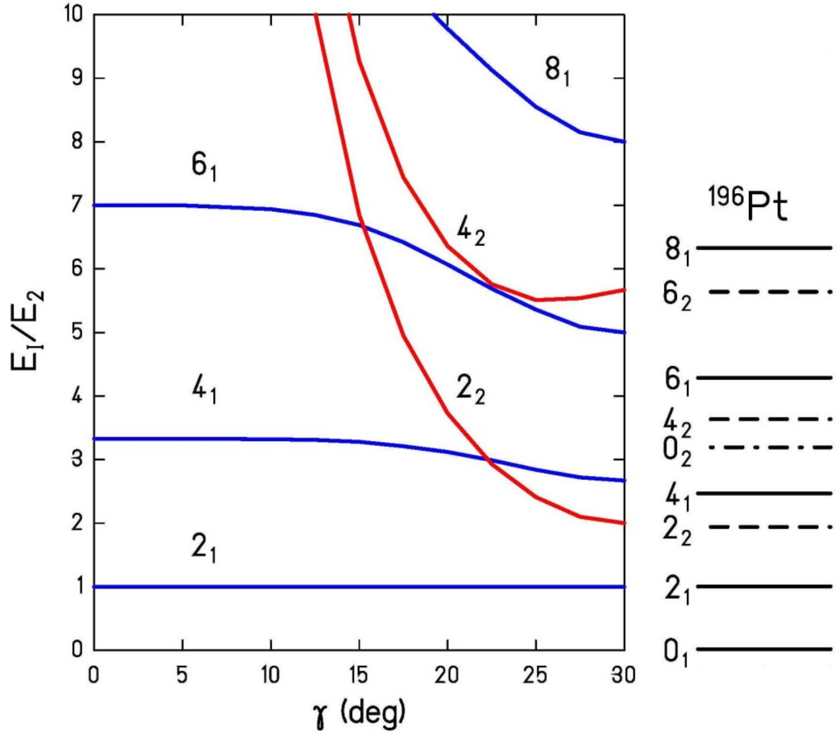


A. Mauthofer et al., Z.Phys. A336, 263 (1990)

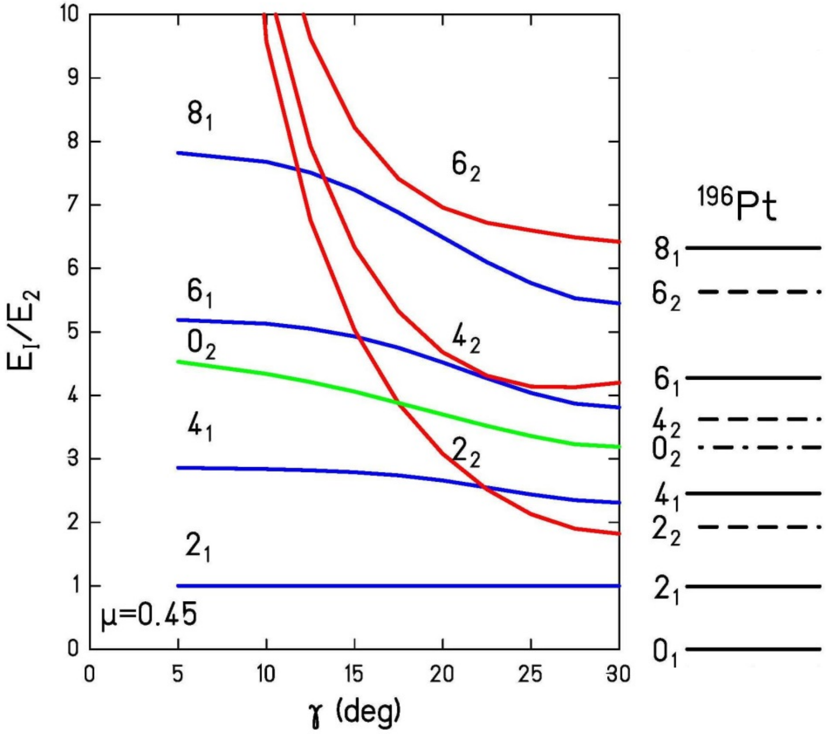
Level scheme of ^{196}Pt



Comparison with
rigid asymmetric rotor model



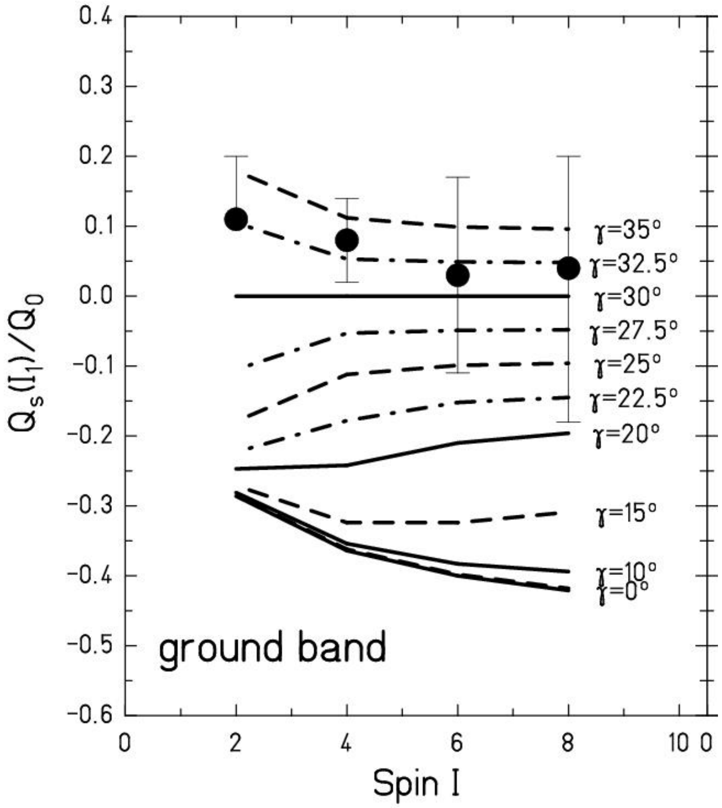
Comparison with
soft asymmetric rotor model



Spin dependence of the spectroscopic quadrupole moment

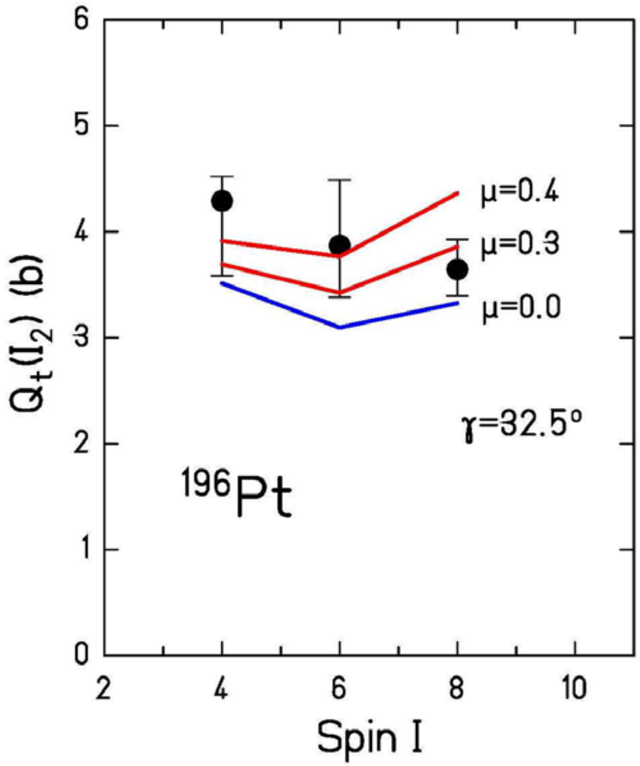
$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I\cdot(2I-1)}{(I+1)\cdot(2I+1)\cdot(2I+3)}}\frac{\langle I\|M(E2)\|I\rangle}{\langle 2_1\|M(E2)\|0_1\rangle}$$

$Q_0 = 3.87(7)\ [b]$ $\qquad Q_0 = \frac{3\cdot Z\cdot R_0^2}{\sqrt{5\pi}}\cdot\beta$ $\qquad \beta = 0.135(2)$



Transition quadrupole moment in the γ-band

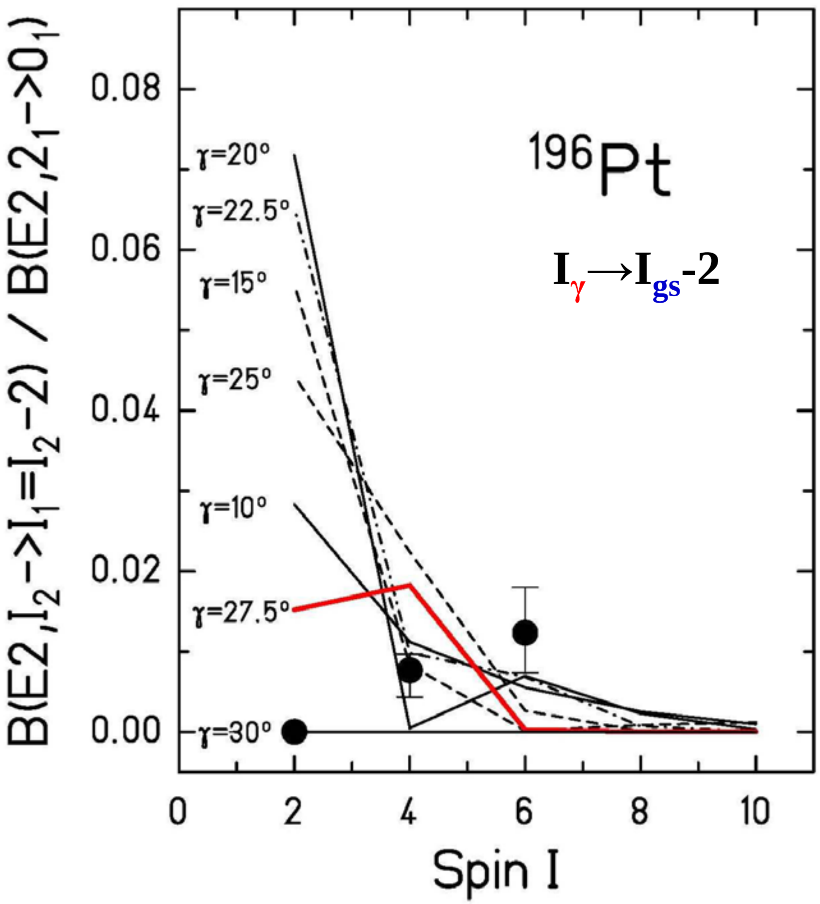
$$Q_t(I_2) = \sqrt{\frac{(2I-2)\cdot(2I-1)\cdot I}{3\cdot(I+1)\cdot(I+2)\cdot(I-2)\cdot(I-3)}} \cdot \sqrt{\frac{16\pi}{5}} \cdot \langle I_2 - 2 \| M(E2) \| I_2 \rangle$$

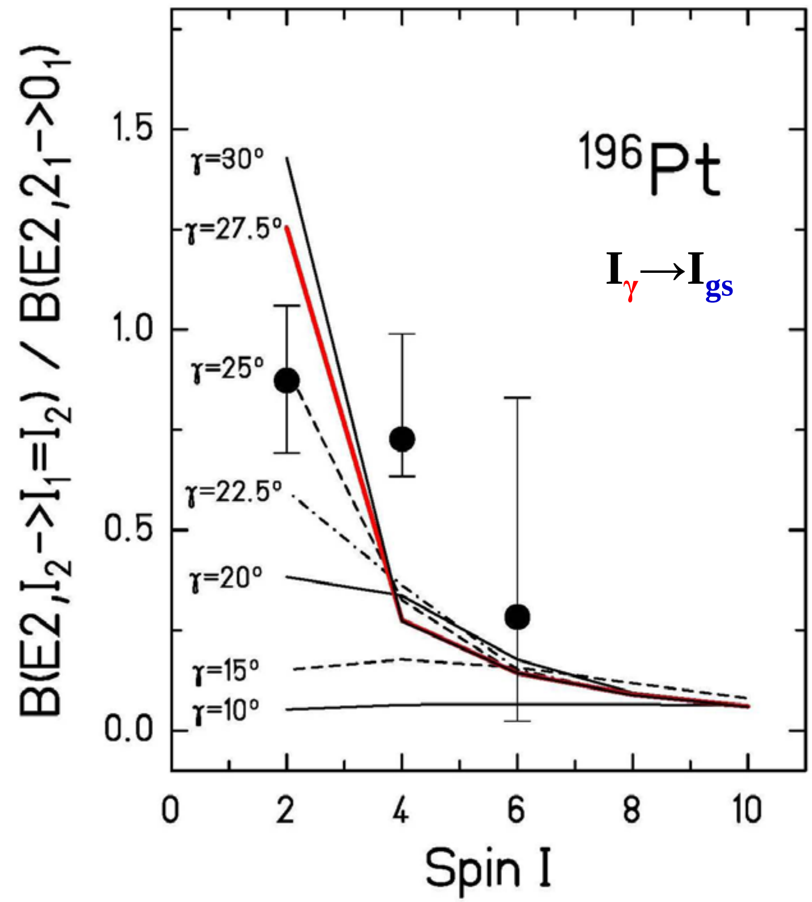


soft ARM

rigid ARM

B(E2)-values connecting the γ- and gs-band





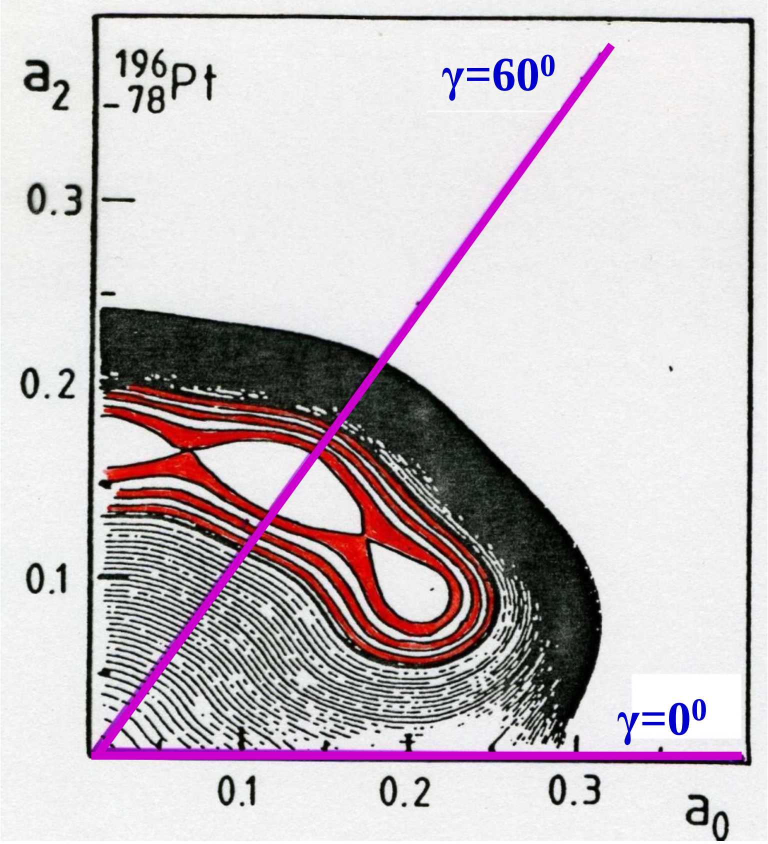
$$\frac{B(E2;2_2 \to 0)}{B(E2;2_1 \to 0)} = \frac{1 - \dfrac{3 - 2\sin^2(3\gamma)}{\sqrt{9 - 8\sin^2(3\gamma)}}}{1 + \dfrac{3 - 2\sin^2(3\gamma)}{\sqrt{9 - 8\sin^2(3\gamma)}}}$$

$$\frac{B(E2;2_2 \to 2_1)}{B(E2;2_1 \to 0)} = \frac{\dfrac{20}{7}\,\dfrac{\sin^2(3\gamma)}{9 - 8\sin^2(3\gamma)}}{1 + \dfrac{3 - 2\sin^2(3\gamma)}{\sqrt{9 - 8\sin^2(3\gamma)}}}$$

Interpretation of the collective properties in ^{196}Pt

14 energy levels and **22** E2 matrix elements can be described by the soft asymmetric rotor model assuming the following parameters:

$$\hbar^2/2\Im = 40.2\ keV \qquad \beta = 0.135 \qquad \gamma = 32.5^0 \qquad \mu = 0.35$$

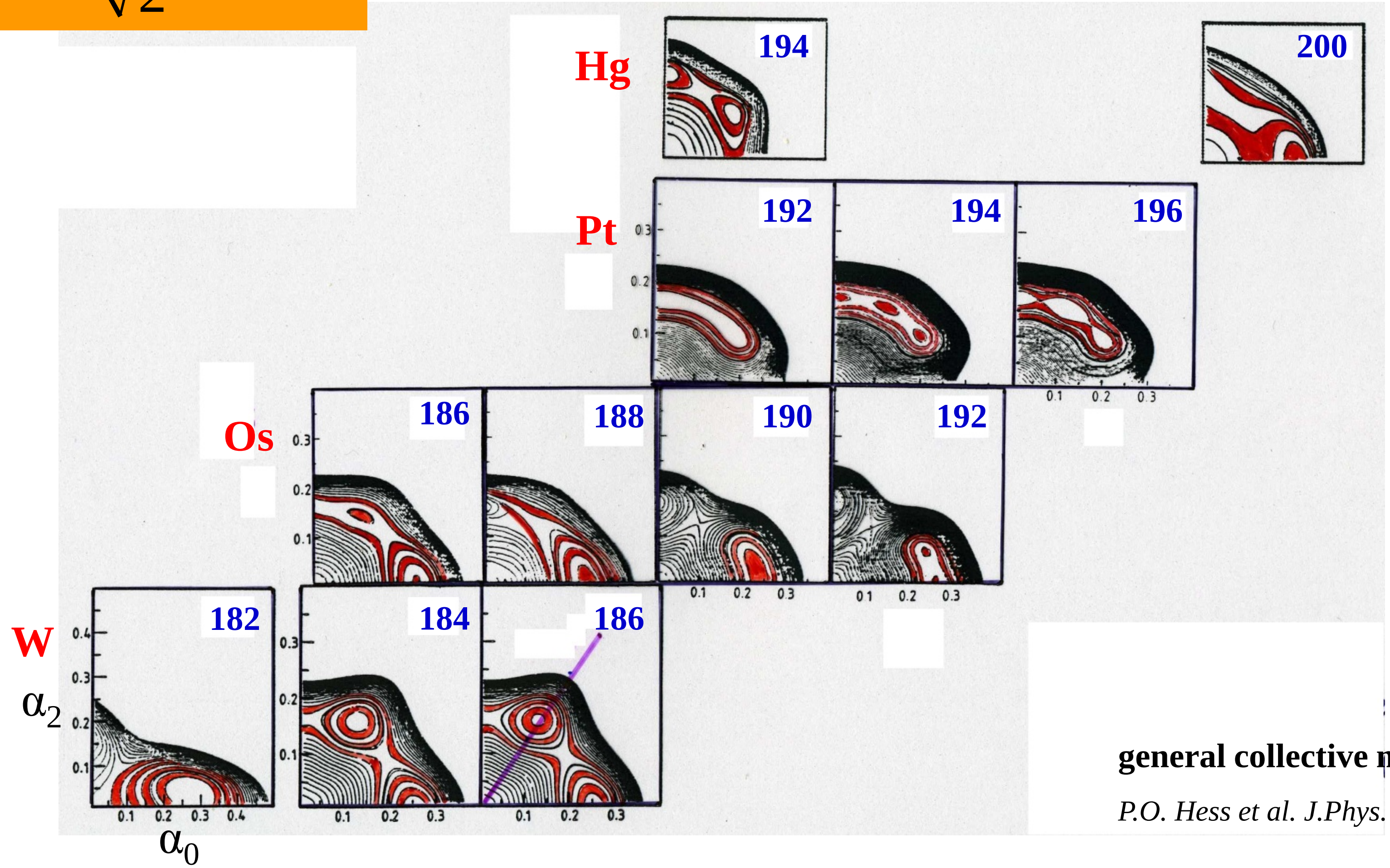


$$a_0 = \beta \cdot \cos\gamma$$
$$a_2 = \frac{1}{\sqrt{2}}\beta \cdot \sin\gamma$$

P.O. Hess et al. J.Phys. G7 (1981), 737

$$a_0 = \beta \cdot \cos\gamma$$

$$a_2 = \frac{1}{\sqrt{2}} \beta \cdot \sin\gamma$$

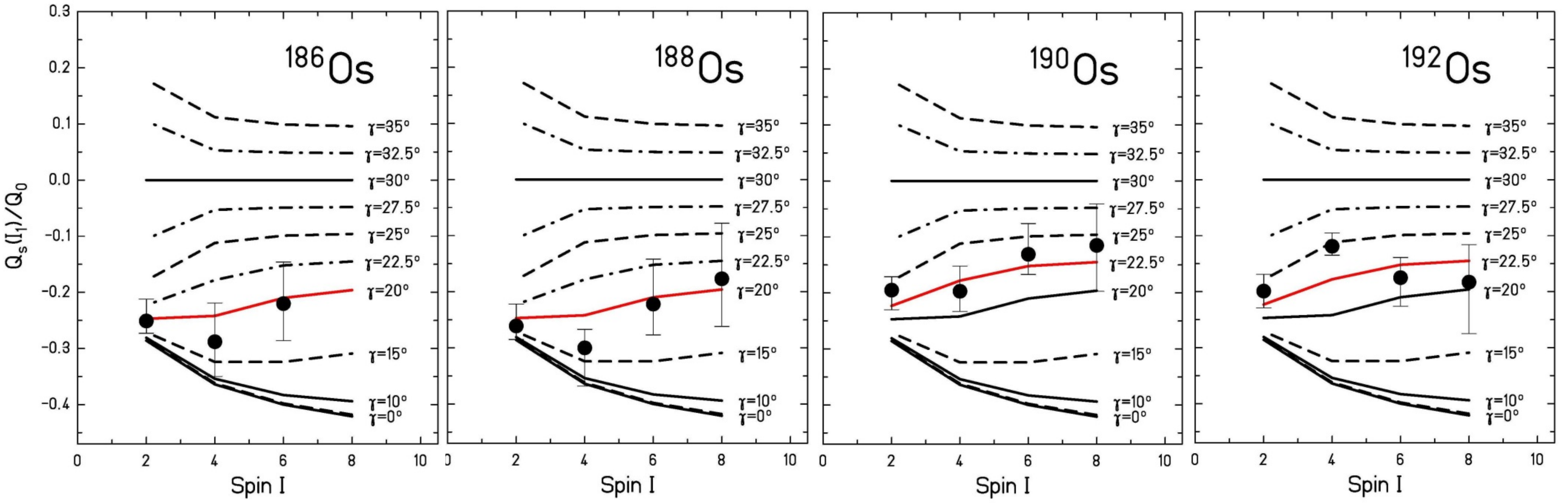


general collective model

P.O. Hess et al. J.Phys. G7 (1981), 737

Spectroscopic quadrupole moments in the ground state band of Os-isotopes

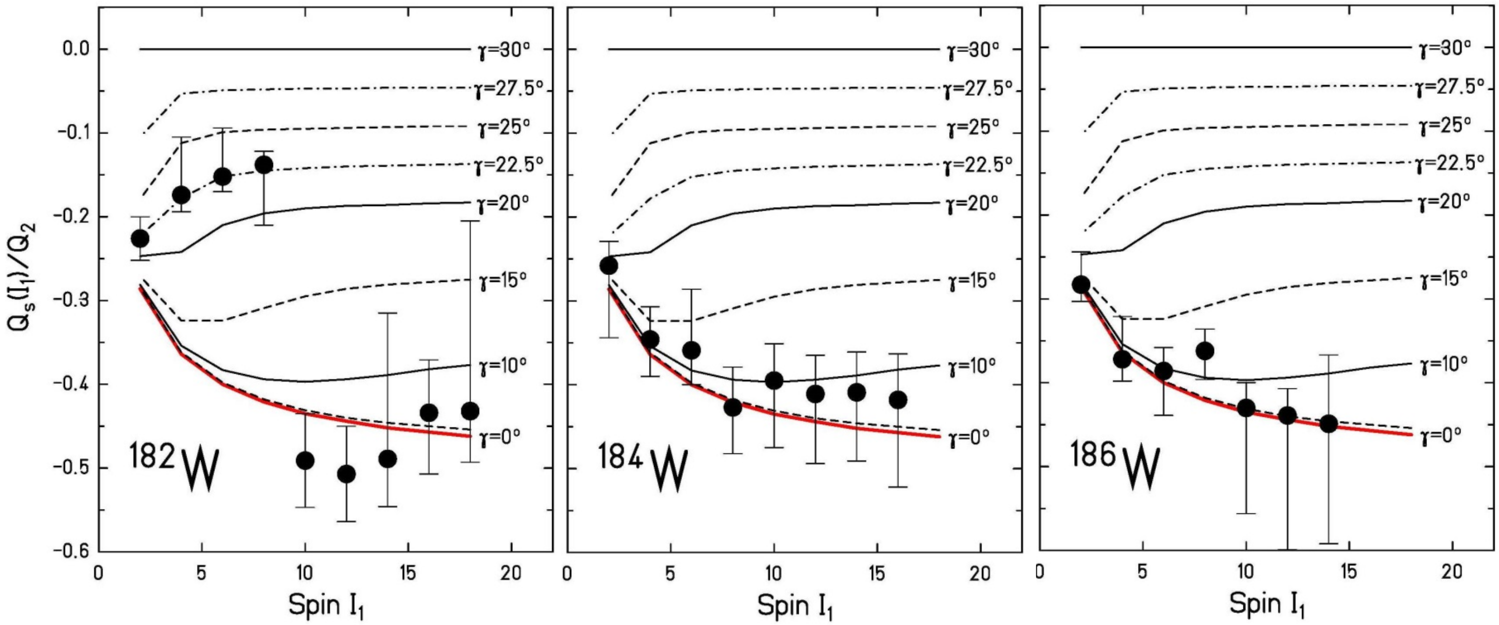
$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I\cdot(2I-1)}{(I+1)\cdot(2I+1)\cdot(2I+3)}}\cdot\frac{\langle I\|M(E2)\|I\rangle}{\langle 2_1\|M(E2)\|0_1\rangle}$$



C.Y. Wu, D. Cline et al.; Ann. Rev. Nucl. Part Sci 36 (1986), 683

Spectroscopic quadrupole moments in the ground state band of W-isotopes

$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I\cdot(2I-1)}{(I+1)\cdot(2I+1)\cdot(2I+3)}}\cdot\frac{\langle I\|M(E2)\|I\rangle}{\langle 2_1\|M(E2)\|0_1\rangle}$$



Q_0 = 7.10 (38) b
β = 0.274 – 0.193

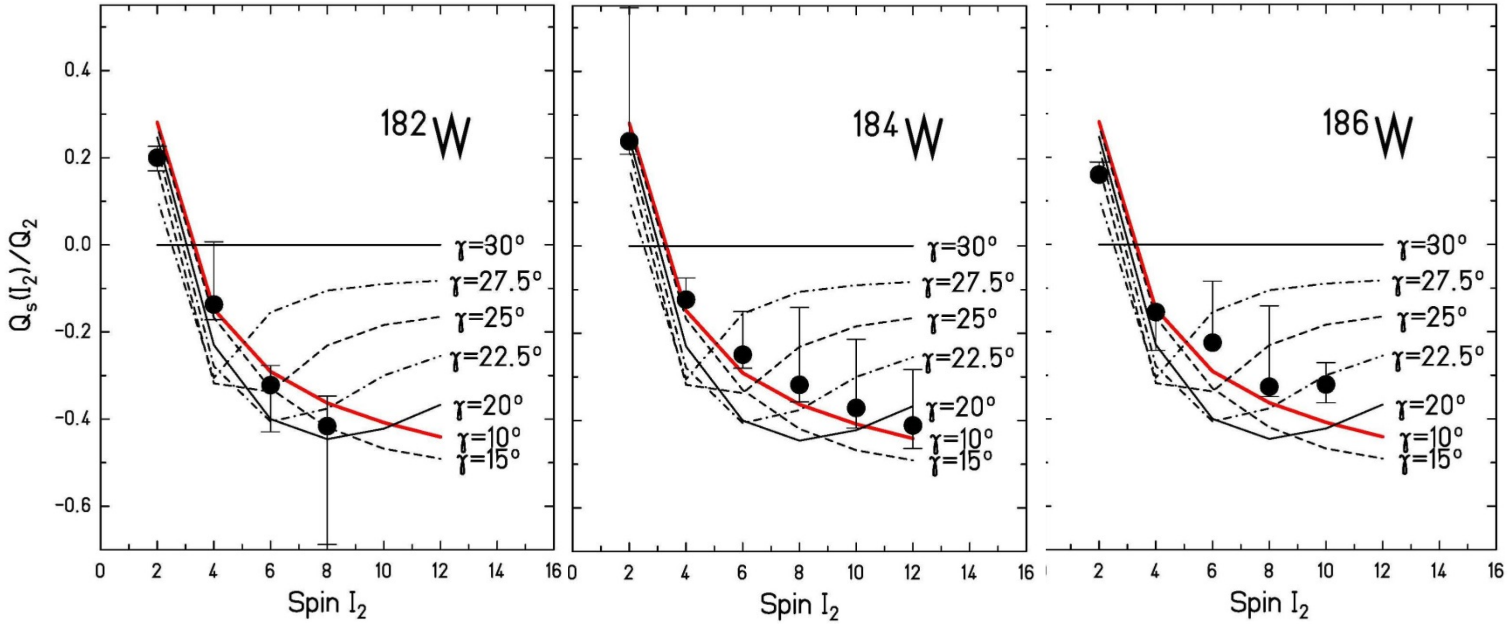
Q_0 = 6.72 (35) b
β = 0.258

Q_0 = 5.87 (29) b
β = 0.223

R. Kulessa et al.; Phys. Lett B218 (1989), 421

Spectroscopic quadrupole moments in the gamma band of W-isotopes

$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I\cdot(2I-1)}{(I+1)\cdot(2I+1)\cdot(2I+3)}}\cdot\frac{\langle I\|M(E2)\|I\rangle}{\langle 2_1\|M(E2)\|0_1\rangle}$$



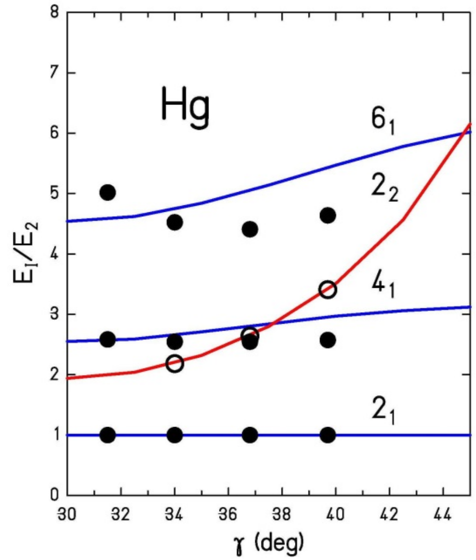
Q_0 = 5.8 (5) b
β = 0.227

Q_0 = 5.6 (3) b
β = 0.219

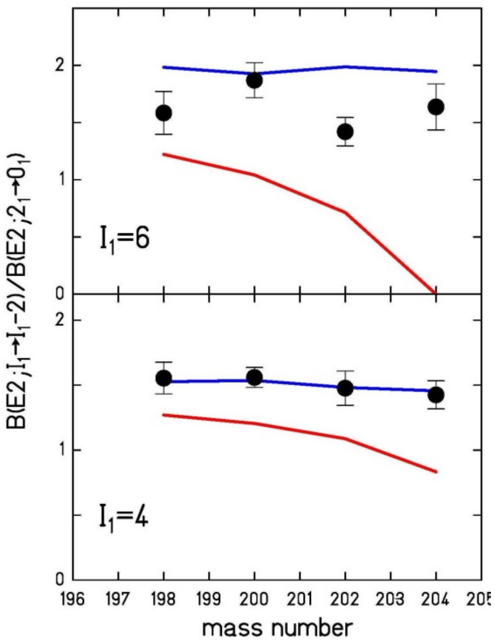
Q_0 = 6.3 (4) b
β = 0.239

R. Kulessa et al.; Phys. Lett B218 (1989), 421

Collective properties in $^{198,200,202,204}Hg$



A	$B(E2;2_1\rightarrow 0_1)$
198	29 spu
200	25 spu
202	17 spu
204	12 spu



soft (μ=0.3) **asymmetric rotor model:**

IBM – O(6) limit:

$$\frac{B(E2;6_1\rightarrow 4_1)}{B(E2;2_1\rightarrow 0_1)}=\frac{5\cdot(N-2)\cdot(N+6)}{3\cdot N\cdot(N+4)}$$

$$\frac{B(E2;4_1\rightarrow 2_1)}{B(E2;2_1\rightarrow 0_1)}=\frac{10\cdot(N-1)\cdot(N+5)}{7\cdot N\cdot(N+4)}$$

boson number **N = 5** to **2**
for ***Hg-isotopes*** with **A=198** to **204**

C. Günther et al.; Z. Phys. A301 (1981), 119
Y.K. Agarwal et al.; Z. Phys. A320 (1985), 295

Parameter of the asymmetric rotor model

isotope	β	γ	γ	μ
^{182}W	0.274	11.4^0	11.2^0	0.17
^{184}W	0.258	13.8^0	13.7^0	0.15
^{186}W	0.223	15.9^0	15.8^0	0.05
^{186}Os	0.196	16.5^0	16.1^0	0.26
^{188}Os	0.185	19.2^0	18.8^0	0.26
^{190}Os	0.184	22.3^0	22.0^0	0.26
^{192}Os	0.168	25.2^0	25.2^0	0.10
^{192}Pt	0.146	-	32.5^0	0.35
^{194}Pt	0.134	-	32.5^0	0.35
^{196}Pt	0.135	-	32.5^0	0.37
^{198}Hg	0.106	36.3^0	38.0^0	0.44
^{200}Hg	0.098	39.1^0	41.0^0	0.44
^{202}Hg	0.082	33.4^0	34.4^0	0.35
^{204}Hg	0.068	31.5^0	31.5^0	0.19

$$B(E2;0_1 \to 2_1) = \frac{5}{16\pi} \cdot Q_0^2 e^2 \cdot \frac{1}{2} \cdot \left[1 + \frac{3 - 2\cdot\sin^2(3\gamma)}{\sqrt{9 - 8\cdot\sin^2(3\gamma)}}\right]$$

$$Q_0 = \frac{3\cdot Z\cdot R_0^2}{\sqrt{5\pi}} \cdot \beta$$

$$\frac{E(2_2)}{E(2_1)} = \frac{3 + \sqrt{9 - 8\cdot\sin^2(3\gamma)}}{3 - \sqrt{9 - 8\cdot\sin^2(3\gamma)}}$$