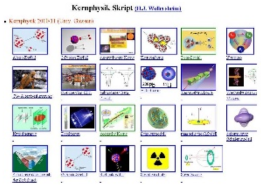
Outline: α-decay

Lecturer: Hans-Jürgen Wollersheim

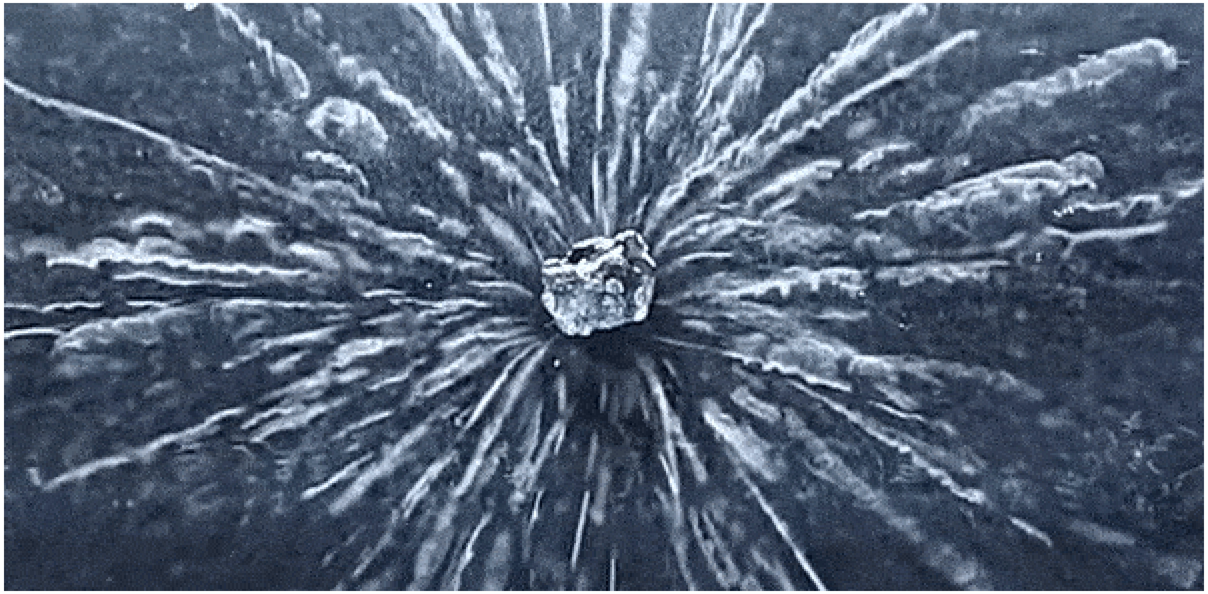
e-mail: h.j.wollersheim@gsi.de

web-page: https://web-docs.gsi.de/~wolle/ and click on



1. energetics of α-decay
2. Geiger – Nuttall law
3. quantum tunneling
4. Gamow factor
5. angular momentum in α-decay

α-decay



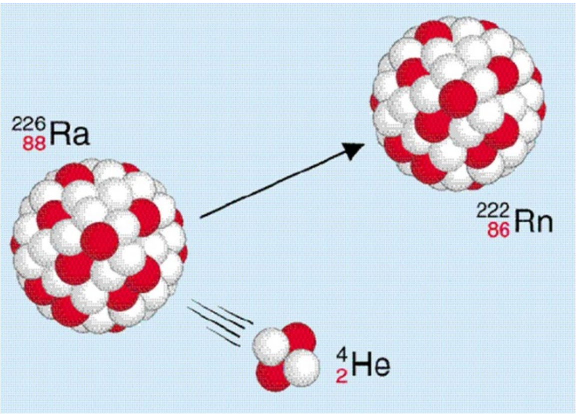
Uranium emitting alpha-particles

cloud chamber

Why α-decay occurs?

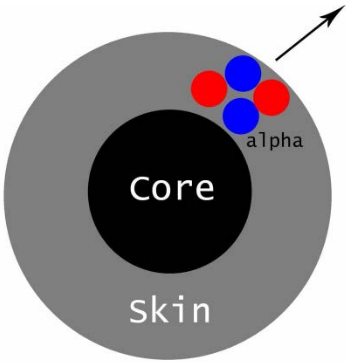
The mass excess = M(A,Z) - A·M(u) (in MeV/c^2) of ^{4}He and its near neighbors

↓N, Z →	0	1	2	3	4
0	-	7.289	-	-	-
1	8.071	13.136	14.931	25.320	37.996
2	-	14.950	**2.425**	11.679	18.374
3	-	25.928	11.386	14.086	15.769
4	-	36.834	17.594	14.908	4.942



Compared to its low-A neighbors in the chart of nuclides, ^{4}He is bound very strongly.

Think classically, occasionally 2 protons and 2 neutrons appear together at the edge of a nucleus, with outward pointing momentum, and bang against the Coulomb barrier.

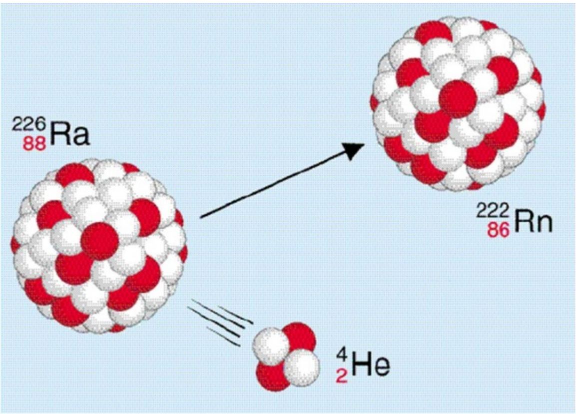


The energetics of α-decay

The α-decay process is determined by the rest mass difference of the initial state and final state.

$$Q = m(A,Z) - m(A-4,Z-2) - m({}^{4}_{2}He)$$

$$Q = BE(A-4,Z-2) + B_{\alpha}(28.3\ MeV) - BE(A,Z)$$



The Q-value of a reaction or of a decay indicates if it happens spontaneously or if additional energy is needed.

AMDC Atomic Mass Data Center

Mass data:
https://www-nds.iaea.org/amdc/

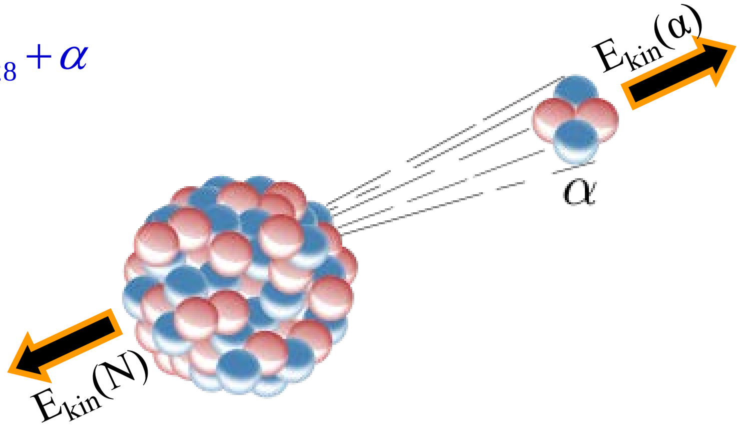
Mass (1u=931.478MeV/c^2)**:**
226.0254u → 222.0176u + 4.0026u
energy gain: 4.87 MeV

Binding energy [M(A,Z) - Z·M(^{1}H) - N·M(^{1}n)] :
-1731.610 MeV → -1708.184 MeV – 28.296 MeV
energy gain: 4.87 MeV

Mass excess [M(A,Z) - A] :
23.662 MeV → 16.367 MeV + 2.425 MeV
energy gain: 4.87 MeV

The Q-value and the kinetic energy of α-particles

What is the α-energy for the system $^{214}_{84}Po_{130} \rightarrow \, ^{210}_{82}Pb_{128} + \alpha$



AMDC Atomic Mass Data Center

Mass data:
https://www-nds.iaea.org/amdc/

BE(^{214}Po) = 1666.0 MeV
BE(^{210}Pb) = 1645.6 MeV
BE(^{4}He) = 28.3 MeV

Q_α = 7.83 MeV

momentum conservation: $m_N \cdot v_N = m_\alpha \cdot v_\alpha \quad \rightarrow \quad v_N = \frac{m_\alpha}{m_N} \cdot v_\alpha$

energy conservation:

$$Q_\alpha = E_{kin}^N + E_{kin}^\alpha$$
$$= \frac{m_N}{2} \cdot v_N^2 + E_{kin}^\alpha$$
$$= \frac{m_N}{2} \cdot \frac{m_\alpha^2}{m_N^2} \cdot v_\alpha^2 + E_{kin}^\alpha$$
$$= \frac{m_\alpha}{m_N} \cdot E_{kin}^\alpha + E_{kin}^\alpha$$
$$= \frac{m_\alpha + m_N}{m_N} \cdot E_{kin}^\alpha$$

$$\rightarrow E_{kin}^\alpha = Q_\alpha \cdot \frac{m_N}{m_N + m_\alpha} = 7.83 MeV \cdot \frac{210}{214} = 7.68\ MeV$$

For a typical α-emitter, the recoil energy is ~100 -150 keV.

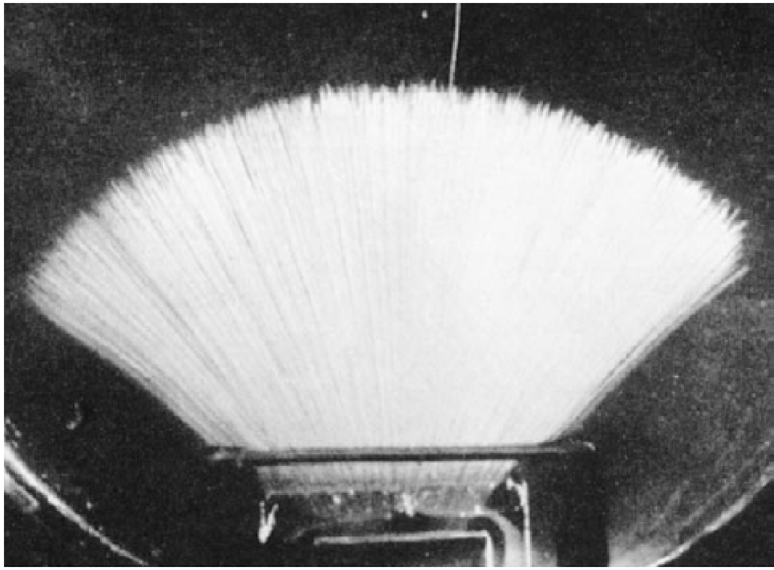
α-decay

Energy differences of atomic masses:

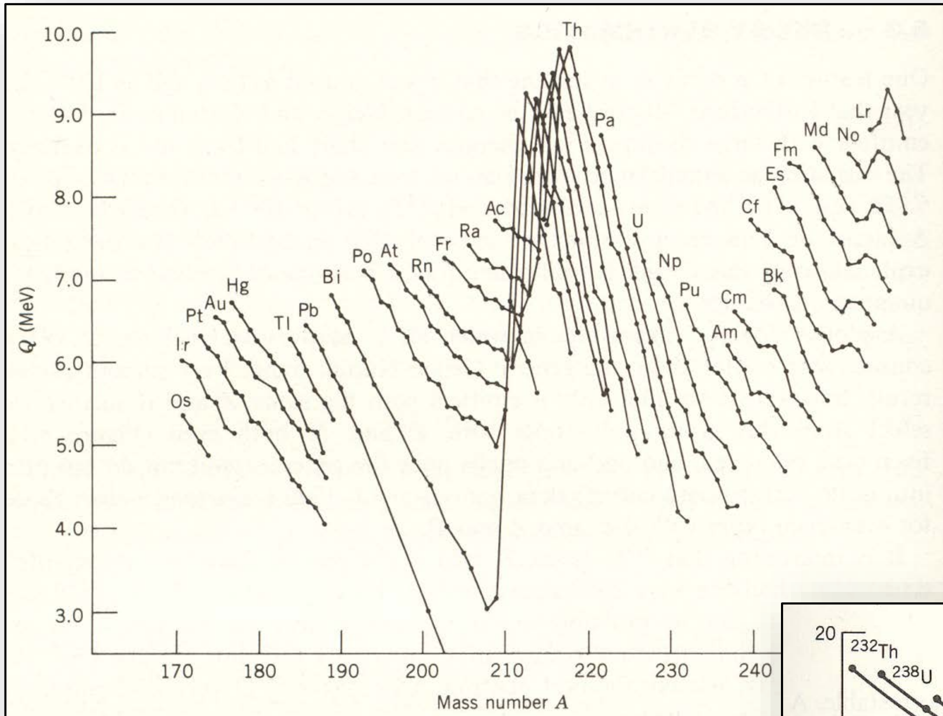
$$Q = m(A,Z) - m(A-4, Z-2) - m({}^{4}_{2}He)$$

$$Q = BE(A-4, Z-2) + B_{\alpha}(28.3\ MeV) - BE(A,Z)$$

The Q-value of the decay is the available energy which is distributed as kinetical energy between the two participating particles. Since the mother and daughter nucleus have fixed masses, the α-particles are mono-energetic.



Tracks of α-particles ($^{214}Po \rightarrow {}^{210}Pb + \alpha$) in a cloud chamber.
The constant length of the tracks shows that the α-particles are mono-energetic (E_{α}=7.7 MeV)



Q-value for α-decay:

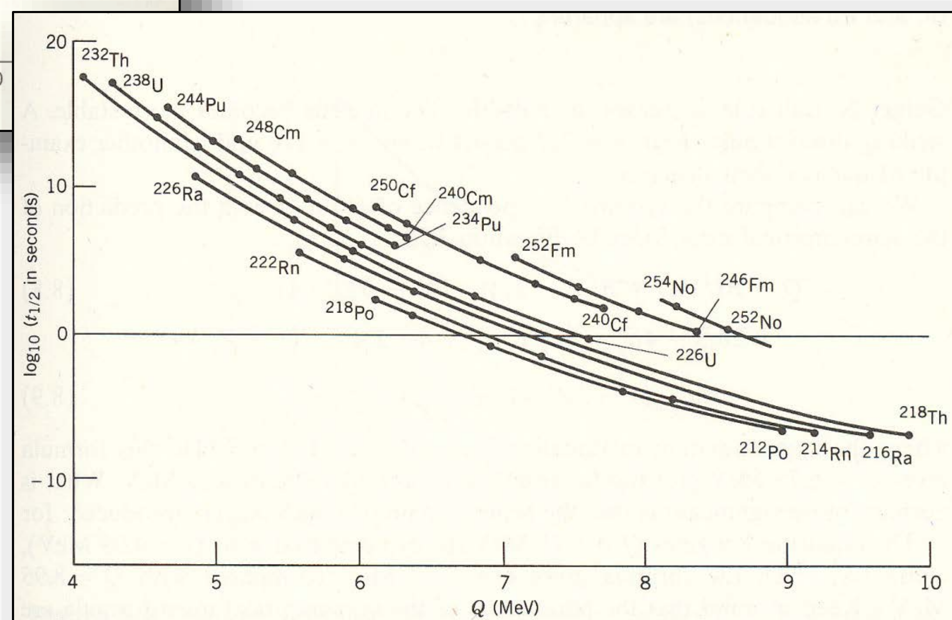
$$Q = BE(A-4, Z-2) + B_{\alpha}(28.3\ MeV) - BE(A,Z)$$

shell effect at N = 126, Z = 82

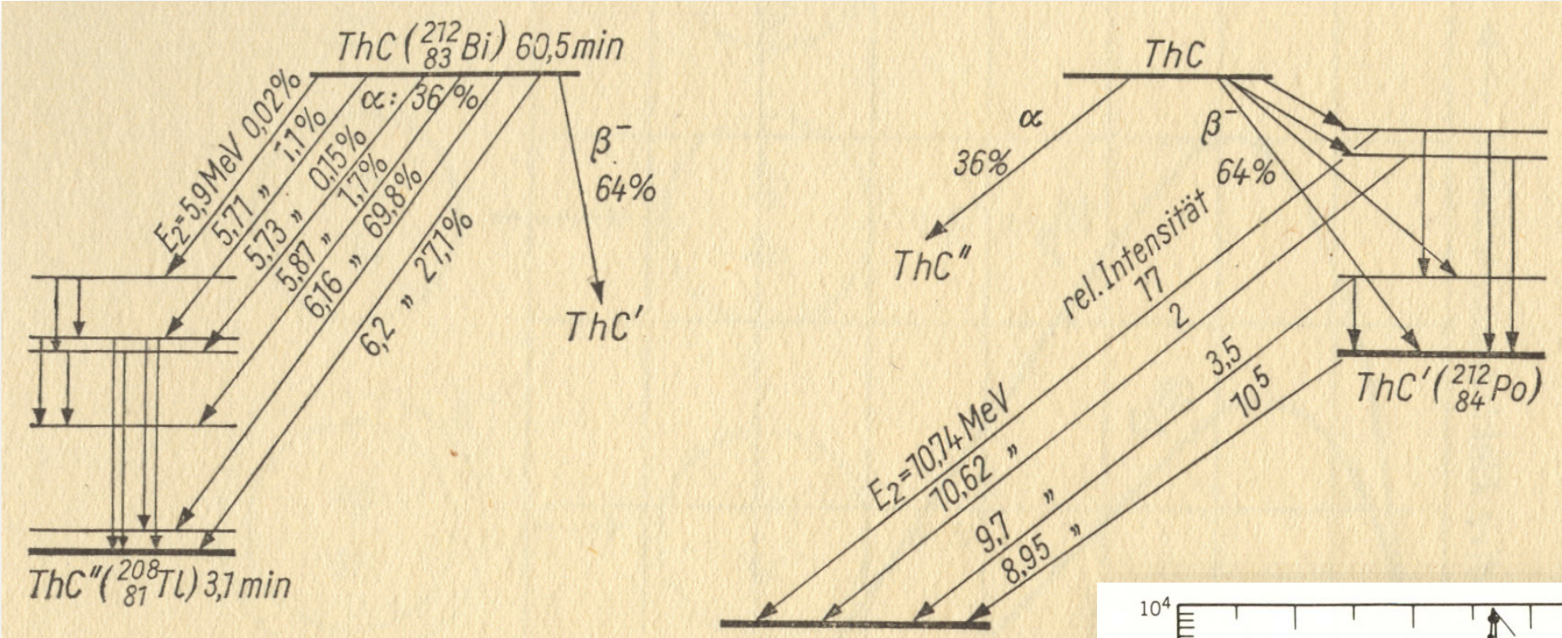
odd-even staggering

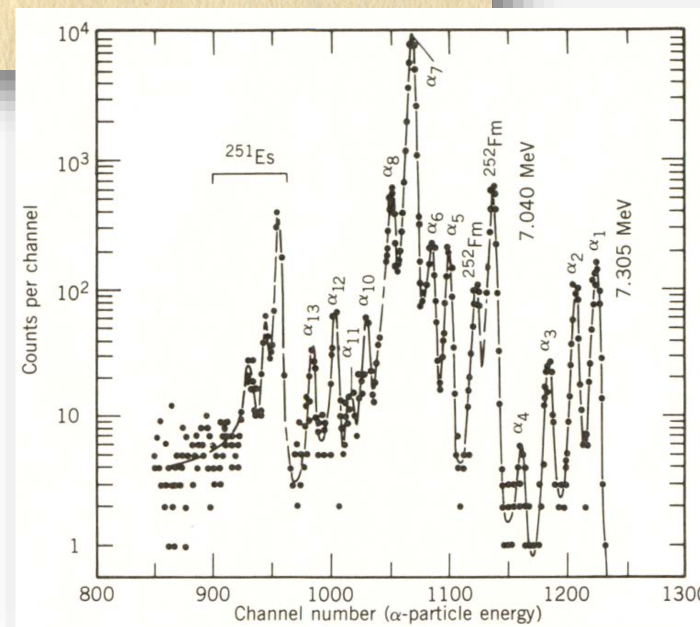
Geiger-Nuttall law:

as the Q-value increases, $T_{1/2}$ decreases



α-decay schemes / spectra



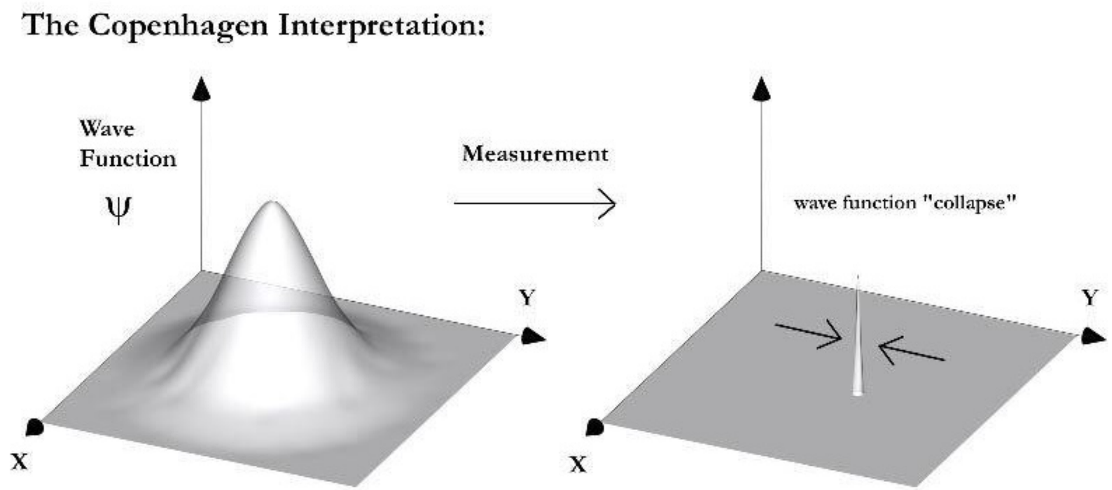


$$^{251}_{100}Fm \rightarrow \, ^{247}_{98}Cf$$

α - spectrum

Brief introduction of quantum mechanics

Richard Feynman: „I think I can safely say that nobody understands quantum mechanics.“



position in space   position in space

<u>When nobody is looking</u>

- System are described by wave functions
- Wave functions obey the Schrödinger equation

<u>When somebody looks</u>

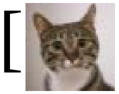
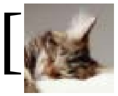
- Wave function collapses to a particular value
- Probability of any outcome is the wave function squared, $p(x) = |\Psi(x)|^2$

Copenhagen interpretation (1927)

REMINDER

Schrödinger's cat

A classical cat is in a definite awake/asleep state:

[cat] = [] or [cat] = []

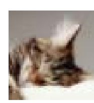
A quantum cat can be in a superposition:

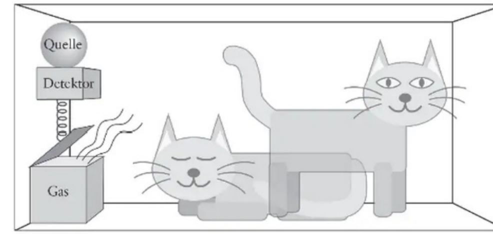
(cat) = ( + )

cat and observer are both quantum:

(cat)(observer) = ( + )()

measurement

(cat,obs) = (,  + , )



Never interfere with each other.

It's as if they have become part of a separate worlds.

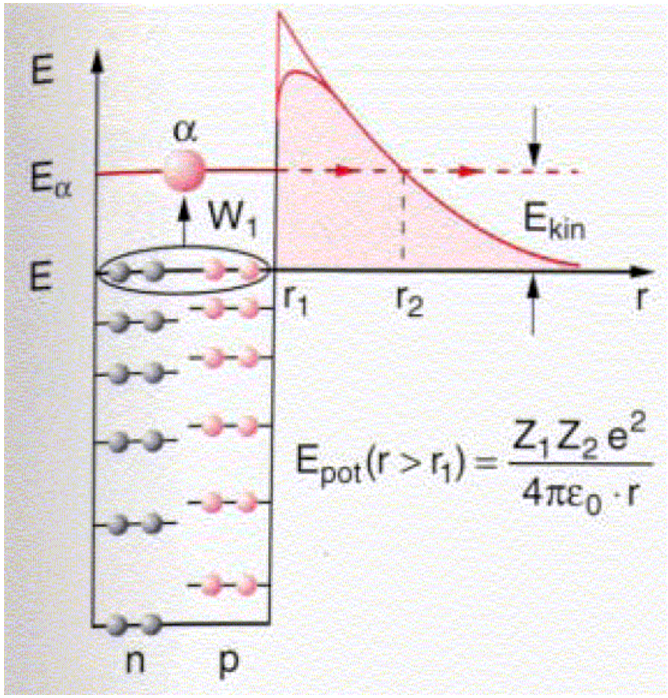
α-decay

Proton and neutrons are bound with up to 7 MeV and can not escape out of the nucleus.
The emission of a bound system is more probable because of the additional binding energy $E_\alpha = 28.3$ MeV.

The Coulomb barrier of the nucleus prevents the α-particles from escaping. The energy needed is generally in the range of about 20-25 MeV.

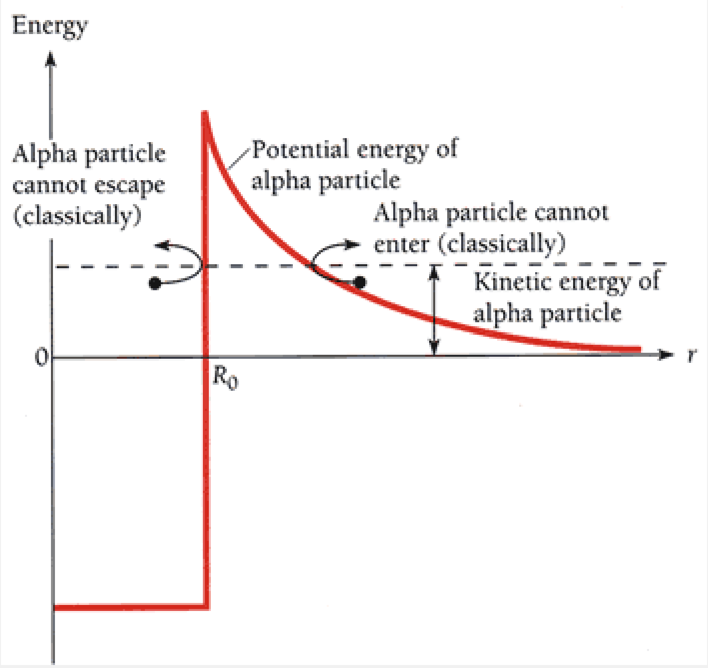
$$V_C = \frac{2 \cdot (Z-2) \cdot e^2}{r} = \frac{2 \cdot 82 \cdot 1.44\ MeV\ fm}{11.25\ fm} = 21\ MeV$$

Classically, the α-particle is reflected on the Coulomb barrier when $E_\alpha < V_C$, but quantum mechanics allows the penetration through the Coulomb potential via the mechanism of tunneling.

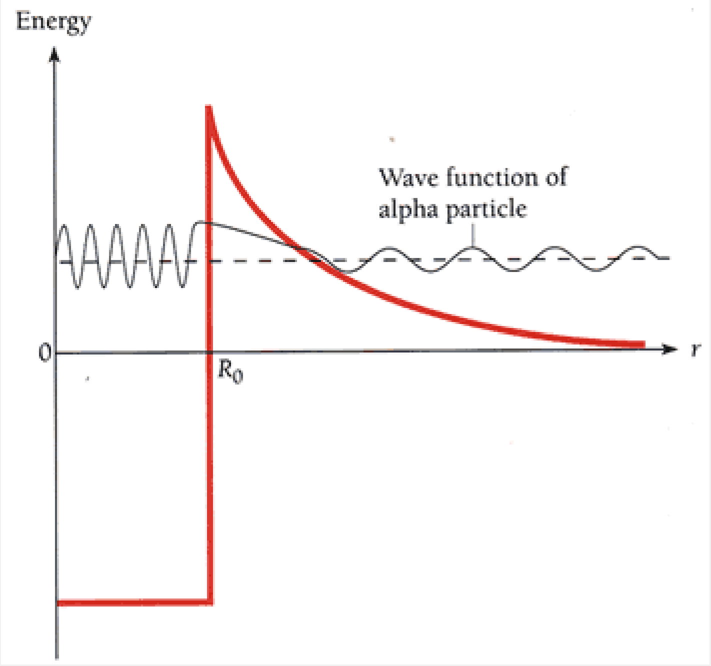


Quantum tunneling and α-decay

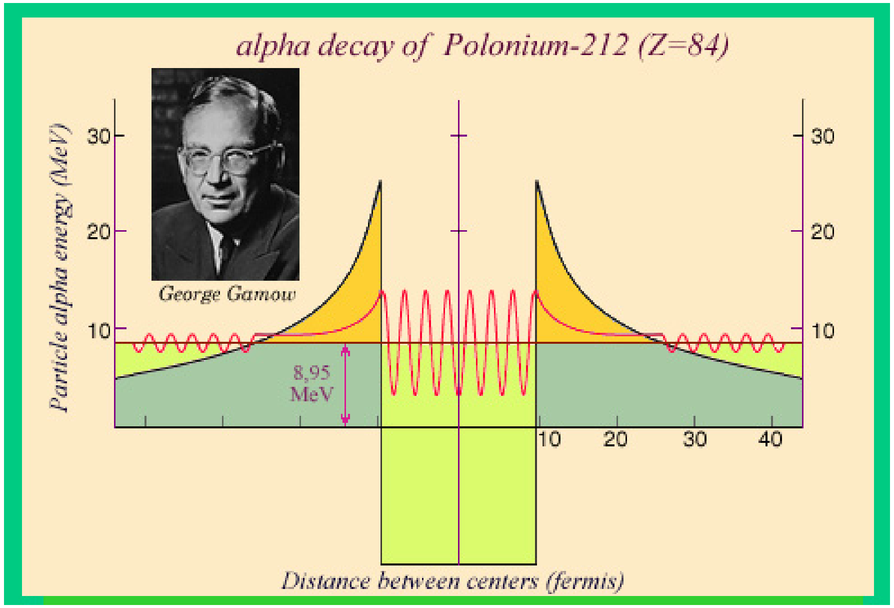
classical treatment



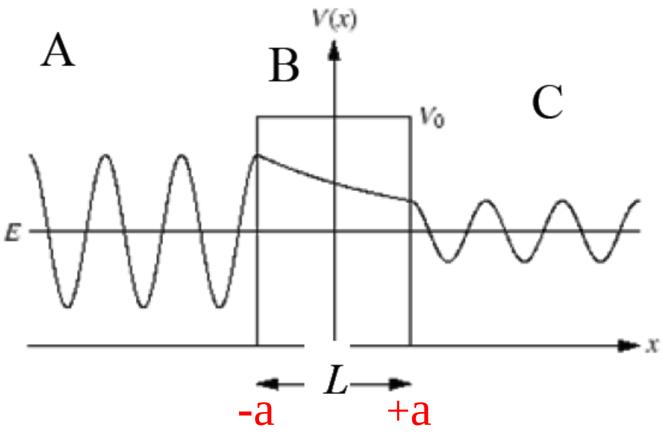
quantum treatment



Quantum tunneling



Intrinsic α-wave function 'leaks' out



general Ansatz for the solutions in area A, B, C

A,C: $\Psi''(x) + k^2 \cdot \Psi(x) = 0; \quad k^2 = 2 \cdot m \cdot E/\hbar^2$

B: $\Psi''(x) + \kappa^2 \cdot \Psi(x) = 0; \quad \kappa^2 = 2 \cdot m \cdot (E - V_0)/\hbar^2$

time independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) + V(x) \cdot \Psi(x) = E \cdot \Psi(x)$$

with $\int |\Psi(x)|^2 = 1$

A: $\Psi(x) = A_1 \cdot e^{i \cdot k \cdot x} + A_2 \cdot e^{-i \cdot k \cdot x}$

C: $\Psi(x) = C_1 \cdot e^{i \cdot k \cdot x} + C_2 \cdot e^{-i \cdot k \cdot x}$

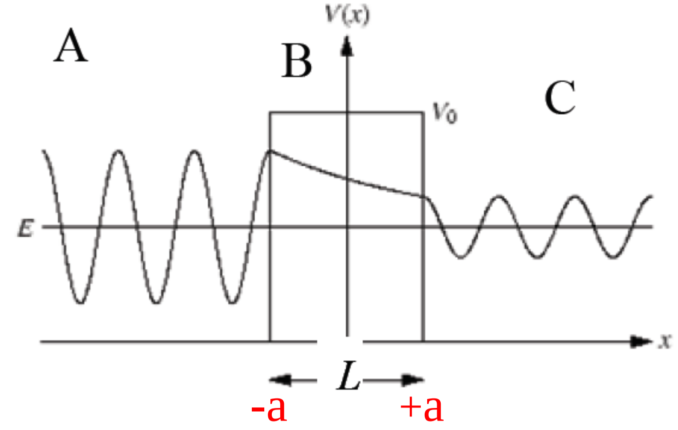
B: $\Psi(x) = B_1 \cdot e^{\kappa \cdot x} + B_2 \cdot e^{-\kappa \cdot x}; \quad \kappa^2 = 2 \cdot m \cdot (V_0 - E)/\hbar$

Quantum tunneling

The α-wave function from left will either be reflected or transmitted through the potential.

- In region C the wave is only travelling to the right
  → $C_2 = 0$

With 4 equations for the 5 unknowns A_1, A_2, B_1, B_2 and C_1 4 quantities can be determined with respect to e.g. A_1.



Reflection coefficient: $R = \frac{|A_2|^2}{|A_1|^2}$

Transmission coefficient: $T = \frac{|C_1|^2}{|A_1|^2}$

x= -a:

$$A_1 \cdot e^{-i\cdot k\cdot a} + A_2 \cdot e^{i\cdot k\cdot a} = B_1 \cdot e^{-\kappa\cdot a} + B_2 \cdot e^{\kappa\cdot a}$$

$$i\cdot k\cdot A_1 \cdot e^{-i\cdot k\cdot a} - i\cdot k\cdot A_2 \cdot e^{i\cdot k\cdot a} = \kappa\cdot B_1 \cdot e^{-\kappa\cdot a} - \kappa\cdot B_2 \cdot e^{\kappa\cdot a}$$

x= +a:

$$C_1 \cdot e^{i\cdot k\cdot a} = B_1 \cdot e^{\kappa\cdot a} + B_2 \cdot e^{-\kappa\cdot a}$$

$$i\cdot k\cdot C_1 \cdot e^{i\cdot k\cdot a} = \kappa\cdot B_1 \cdot e^{\kappa\cdot a} - \kappa\cdot B_2 \cdot e^{-\kappa\cdot a}$$

$$T = \left[1 + \frac{V_0^2}{4\cdot E\cdot (V_0 - E)} \cdot sinh^2 \sqrt{2\cdot m\cdot (V_0 - E)}/\hbar \cdot L\right]^{-1}$$

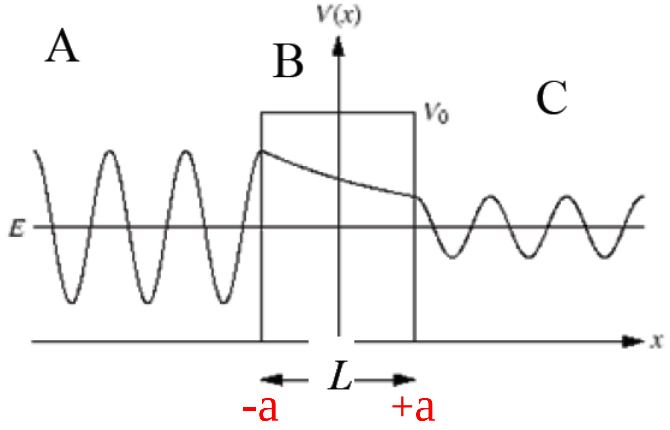
$$T \approx \frac{16\cdot E\cdot (V_0 - E)}{V_0^2} \cdot exp\left\{-2\cdot \sqrt{2\cdot m\cdot (V_0 - E)}\frac{L}{\hbar}\right\}; \quad E \ll V_0$$

$$sinh^2 x = \frac{1}{4}\cdot (e^{2x} + e^{-2x}) - 1/2$$

Quantum tunneling

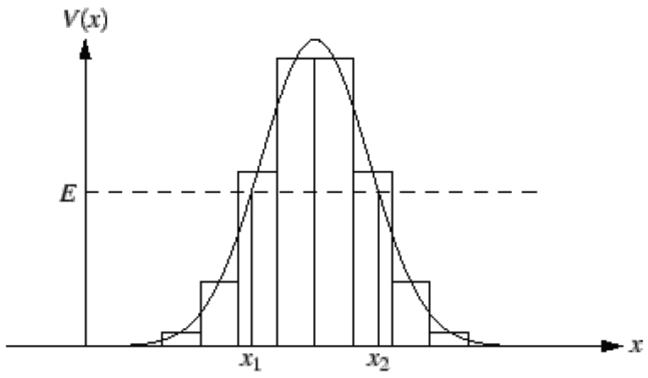
For a simple box potential one obtains the exact solution for transmission coefficient:

$$T(E) = exp\left\{-\sqrt{2 \cdot m \cdot (V_0 - E)}\,\frac{2 \cdot L}{\hbar}\right\}$$



In case of a more realistic potential (see below) one has to use step functions (Δx width) as an approximation

$$T(E) = exp\left\{-2\int_{x_i}^{x_a} dx\,\left(\frac{1}{\hbar}\cdot\sqrt{2 \cdot m \cdot [V(x) - E]}\right)\right\}$$



Gamow factor

probability for tunneling: $T(E) = e^{-2\cdot G}$

with Gamow factor G: $$G(E_\alpha) = \int_{R_i}^{R_a} dr \left(\frac{1}{\hbar} \sqrt{2m(V(r) - E)} \right)$$

$$V(R_a) = E_\alpha = \frac{2Ze^2}{R_a}$$

$$\mathbf{G(E_\alpha)} = \frac{2}{\hbar}\sqrt{2m} \int_{R_i}^{R_a} dr \sqrt{\frac{2Ze^2}{r} - E}$$

$$= \frac{2}{\hbar}\sqrt{\frac{2m}{\mathbf{E}}}\, 2Ze^2 \left\{ arccos\sqrt{\frac{R_i}{R_a}} - \sqrt{\frac{R_i}{R_a} - \frac{R_i^2}{R_a^2}} \right\}$$

notice $\frac{R_i}{R_a} = \frac{E_\alpha}{V_C}$ *for thick barrier* $E_\alpha \ll V_C$ or $R_a \gg R_i$ $\left\{ arcos\sqrt{\frac{R_i}{R_a}} - \sqrt{\frac{R_i}{R_a} - \frac{R_i^2}{R_a^2}} \right\} \approx \frac{\pi}{2} - \sqrt{\frac{R_i}{R_a}}$

α-decay systematics: Geiger-Nuttall law

❖ Geiger-Nuttall law: Relation between the half-life and the energy of α-particles

Ansatz for α-decay probability λ [s^{-1}]:

$$\lambda = S \cdot \omega \cdot P$$

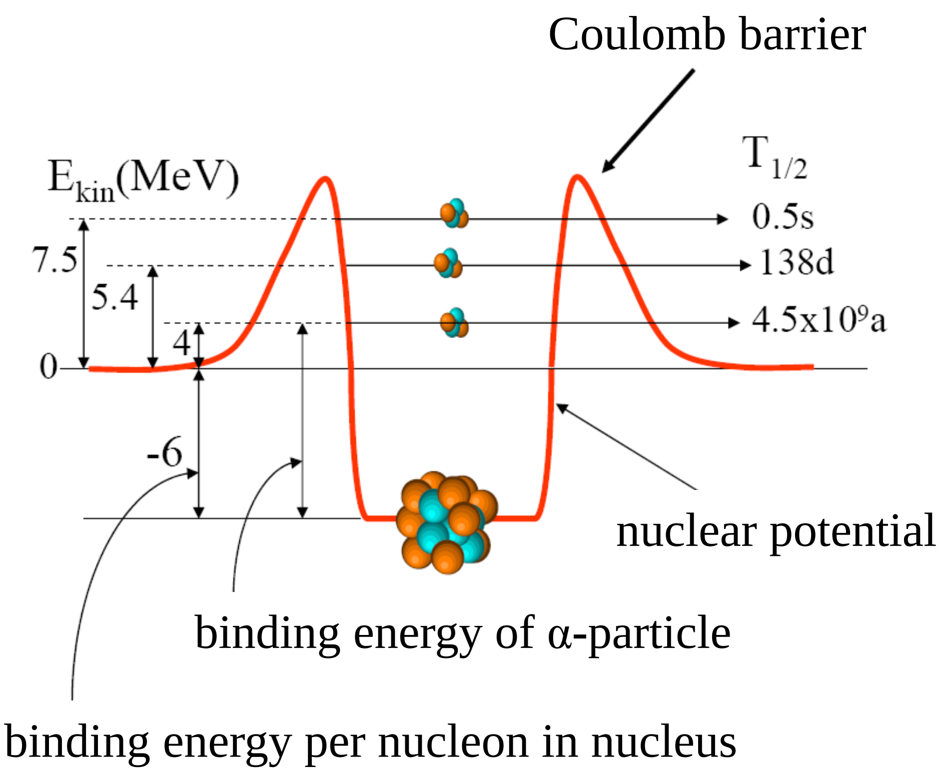
- **S** is the probability that an α-particle has been formed inside of the nucleus
- **ω** is the frequency of the α-particle hitting the Coulomb barrier V_0, where the velocity v is given by the kinetic energy and R is the nuclear radius

$$\omega = \frac{1}{\Delta t} = \frac{v}{2R} = \frac{\sqrt{2V_0/m_\alpha}}{2R} = \frac{\sqrt{2 \cdot \frac{50MeVc^2}{3727MeV}}}{2 \cdot 10fm} \sim 2 \cdot 10^{21}\ s^{-1}$$

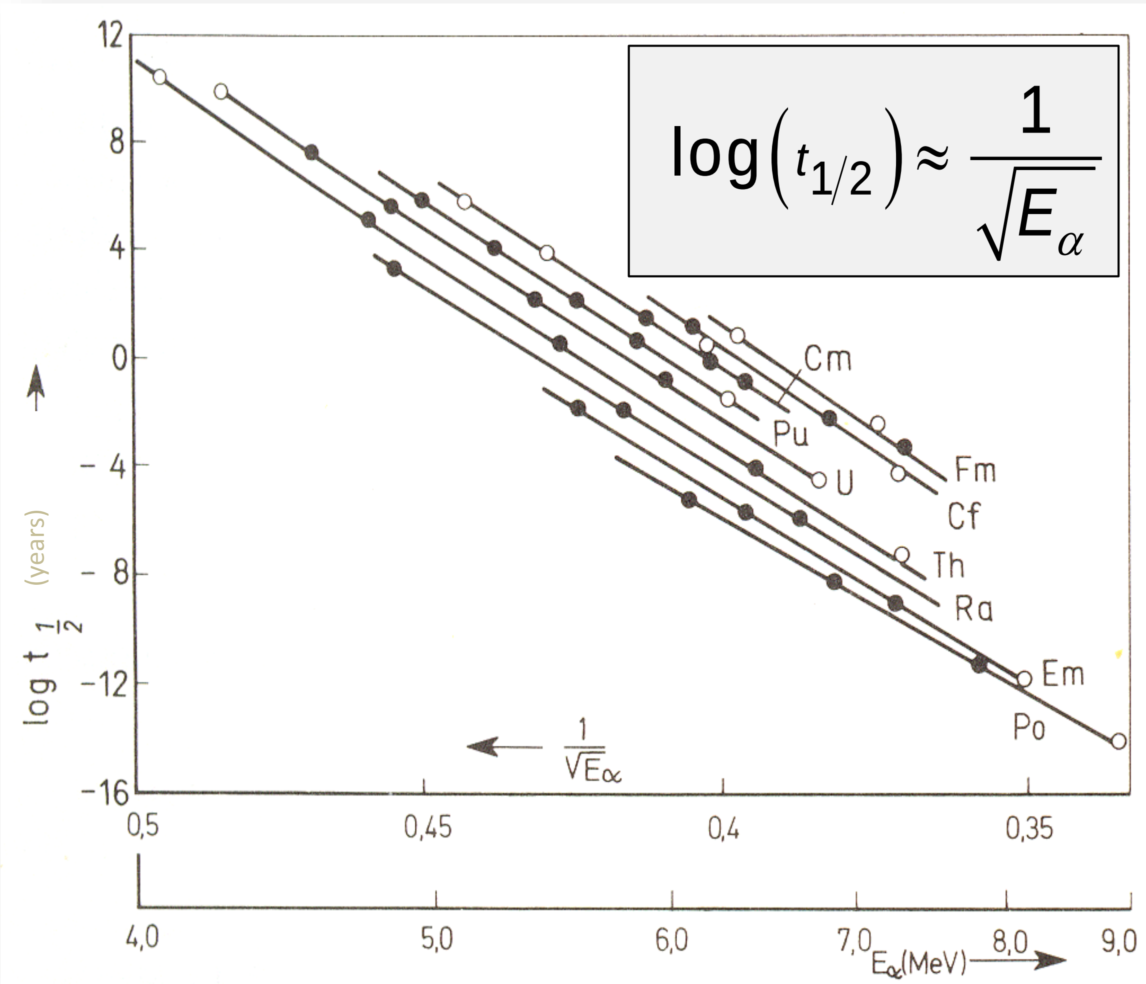
- **P** = T(E) = e^{-2G} is the probability of the tunneling process, where G is the Gamow factor

$$ln\lambda = lnS - ln\Delta t - 2G(E_\alpha)$$

$$= b(Z) - a\frac{Z}{\sqrt{E_\alpha}} \approx -a\frac{Z}{\sqrt{E_\alpha}}$$



α-decay systematics: Geiger-Nuttall law



$$\log\left(t_{1/2}\right) \approx \frac{1}{\sqrt{E_\alpha}}$$

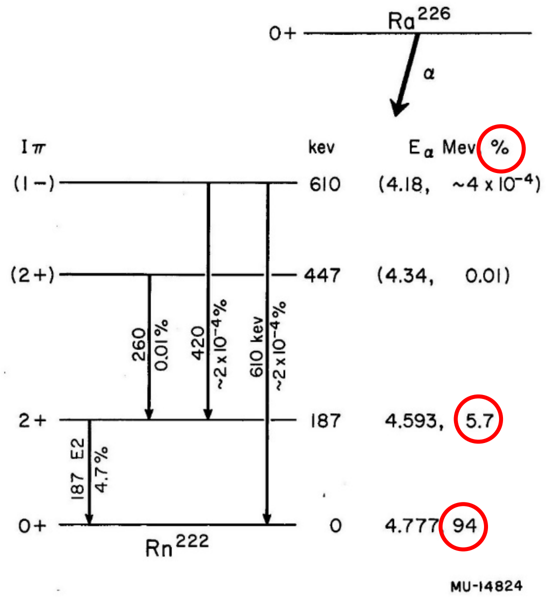
Angular momentum in α-decay

If the α-particle carries off angular momentum, we must add the repulsive potential associated with the centrifugal barrier to the Coulomb potential, $V_C(r)$:

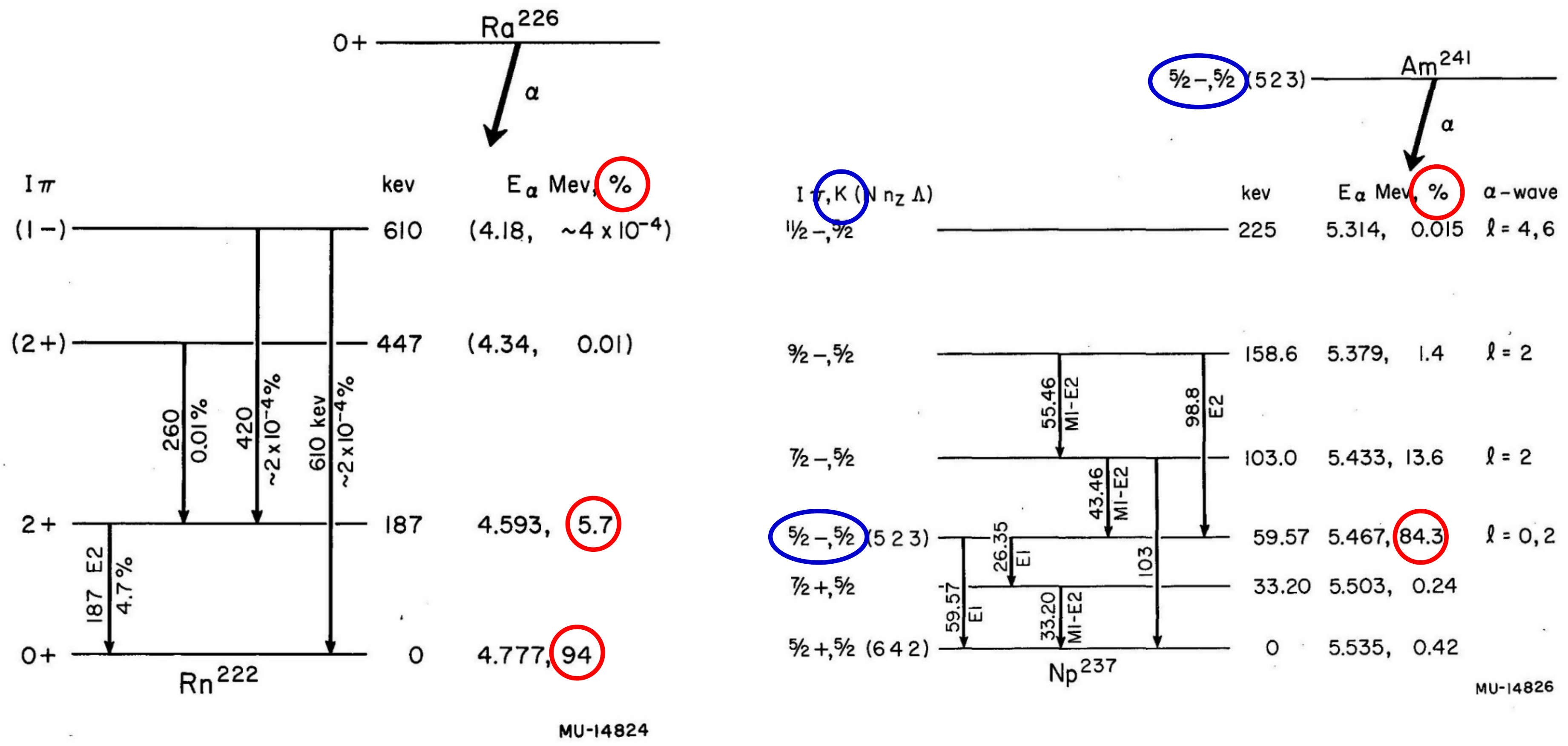
$$V(r) = V_C(r) + \frac{\ell(\ell+1)\hbar^2}{2m_\alpha r^2}$$

The effect on $_{90}$Th with Q=4.5 MeV is:

ℓ	0	1	2	3	4	5	6
T_ℓ/T_0	1	0.84	0.60	0.36	0.18	0.078	0.028



Angular momentum in α-decay

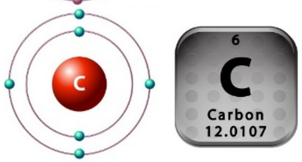


Cluster decay probability

If α decay can occur, surely ^{8}Be and ^{12}C decay can occur as well. It is just a matter of relative probability. For these decays, the escape probabilities are given approximately by:

$$T_{{}^{8}_{4}Be} = T_{\alpha}^{2}$$

$$T_{{}^{12}_{6}C} = T_{\alpha}^{3}$$



$$T_{{}^{A}X} = T_{\alpha}^{Z/2}$$

The last estimate is for a ^{A}X cluster, with Z protons and an atomic mass of A.