The liquid drop model

Introduction to Nuclear Science

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Outline

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1. Total binding energy
2. Neutron separation energy

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3. Proton separation energy

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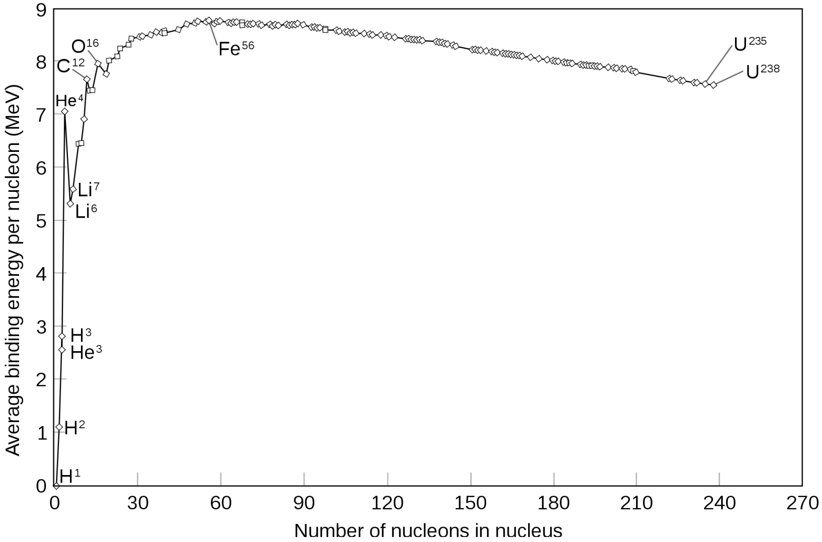
Total binding energy

- Total binding energy B_{tot} is defined as

$$M_a = Z * M(^1H) + (A - Z) * M(n) - \frac{B_{tot}(Z, A)}{c^2}. \qquad (1)$$

- The energy B_{tot} is released in binding Z hydrogen atoms and $N = A - Z$ neutrons into the corresponding nucleus with atomic number Z and mass number A.
- The same amount of energy is needed to separate the nucleus into its constituents: neutron and hydrogen atoms.

Average binding energy per nucleon B_{tot}/A



Average binding energy per nucleon B_{tot}/A

- Average binding energy per nucleon is nearly 8 MeV/nucleon, million times larger compared to chemical binding energies.
- The experimental curve peaks around ^{56}Fe, actually ^{62}Ni is the most bound nucleus.
- Nuclear energy can be released by fission of heavy nuclei, like Uranium. Fission is a process used in reactors.
- Nuclear energy can also be released by fusing light nuclei. This is the process generating energy in stars. Controlled fusion of hydrogen, deuterium and tritium is an active area of research as it offers nearly endless source of energy.

Neutron separation energy

- Neutron separation energy is the work needed to separate a neutron from a nucleus with Z_i, A_i, ending up with the final nucleus of $Z_f = Z_i$ and $A_f = A_i - 1$:

$$^{A_i}_{Z_i}X_{N_i} + S_n \;\rightarrow\; ^{A_i-1}_{Z_i}X_{N_i-1} + n \tag{2}$$

- This work can be calculated from the difference between the mass of initial nucleus and the mass of the final nucleus plus the mass of the neutron.

$$S_n = (m_{Z_f,A_f} + m_{neutron} - m_{Z_i,A_i})c^2. \tag{3}$$

- Negative S_n indicate that the neutron is unbound and the initial nucleus can decay by neutron emission. Lifetime for spontaneous neutron decay is very short 10^{-22} s, thus negative S_n implicates the limit of existence of stable isotopes for a given $Z_i = Z_f$.

Neutron separation energy

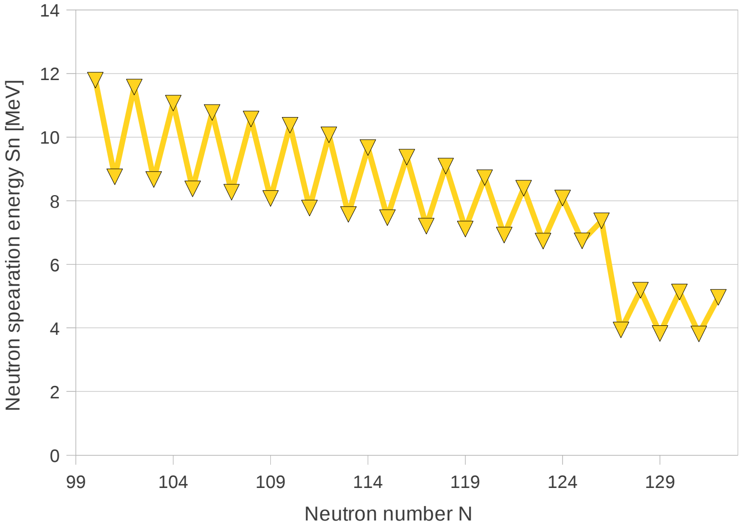
- Using methods outlined in Lecture 1 the separation energy of a neutron can be expressed in terms of the mass excess.

$$S_n = (\Delta M(Z_f, A_f) + \Delta M(0,1) - \Delta M(Z_i, A_i))c^2. \quad (4)$$

- It can also be expressed using total binding energies

$$S_n = B(Z_i, A_i) - B(Z_f, A_f) = B(Z, A_i) - B(Z, Ai - 1). \quad (5)$$

Neutron separation energy in Z=82 Pb isotopes



Neutron separation energy in Z=82 Pb isotopes

- Note that neutron separation energy is larger for isotopes with even number of neutrons as compared to odd number of neutrons.
- This indicates that paired neutrons result in significantly ($\sim$20%) larger total binding per nucleon.
- Also, note a step-like change in neutron separation energy at N=126.
- Such discontinuities indicate closing of neutron shells.

Proton separation energy

- Proton separation energy is the work needed to separate a neutron from a nucleus with Z_i, A_i, ending up with the final nucleus of $Z_f = Z_i - 1$ and $A_f = A_i - 1$:

$$ {}^{A_i}_{Z_i}X_{N_i} + S_p \rightarrow {}^{A_i-1}_{Z_i-1}X_{N_i} + p \tag{6}$$

- This work can be calculated from the difference between the mass of initial nucleus and the mass of the final nucleus plus the mass of the neutron.

$$ S_p = (m_{Z_f,A_f} + m_{proton} - m_{Z_i,A_i})c^2. \tag{7}$$

- Negative S_p indicate that the proton is unbound and the initial nucleus can decay by proton emission. Lifetime for spontaneous proton decay are significantly longer than in case of a neutron decay $\sim 10^{-6}$ s, since unbound protons are held inside a nucleus by the Coulomb barrier (more on that will come).

Neutron separation energy

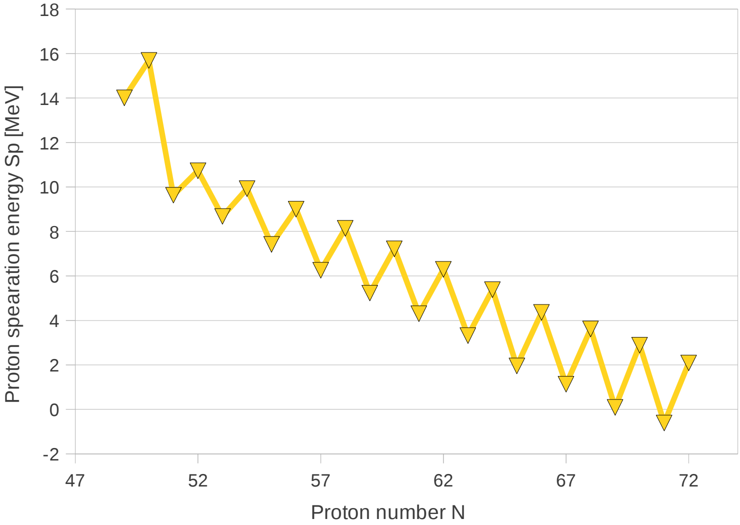
- Using methods outlined in Lecture 1 the separation energy of a proton can be expressed in terms of the mass excess.

$$S_p = (\Delta M(Z_f, A_f) + \Delta M(1,1) - \Delta M(Z_i, A_i))c^2. \quad (8)$$

- It can also be expressed using total binding energies

$$S_p = B(Z_i, A_i) - B(Z_f, A_f) = B(Zi, A_i) - B(Zi - 1, Ai - 1). \quad (9)$$

Proton separation energy in N=82 isotones



Proton separation energy in N=82 isotones

- Note that proton separation energy is larger for isotones with even number of protons as compared to that with odd number of protons.
- This indicates that paired protons result in significantly (∼20% or more) larger total binding per nucleon.
- Also, note a step-like change in proton separation energy at Z=50.
- Such discontinuities indicate closing of protons shells.
- Note that $S_p < 0$ for ^{153}Lu at Z=71, this isotope lives for a short time thanks to the Coulomb barrier hindering proton decay.

Liquid drop model

- In 1935 German physicist Carl Friedrich von Weizsäcker proposed a nuclear model (now called "the liquid drop model") explaining experimental trends observed for average binding energy per nucleon.
- The model assumes five main contributions to the average binding energy, thus also to the mass, of a nucleus:
  1. volume energy
  2. surface energy
  3. Coulomb energy
  4. asymmetry energy
  5. pairing energy
- The volume energy,surface energy and the Coulomb energy terms are analogues to contributions to the energy of a charged liquid drop.
- This analogy justifies the name for the model.

Liquid drop model

- The liquid drop model postulates that

$$B(Z,A) = a_V A - a_S A^{\frac{2}{3}} - a_C \frac{Z^2}{A^{\frac{1}{3}}} - a_A \frac{(N-Z)^2}{A} + \delta(A,Z) \quad (10)$$

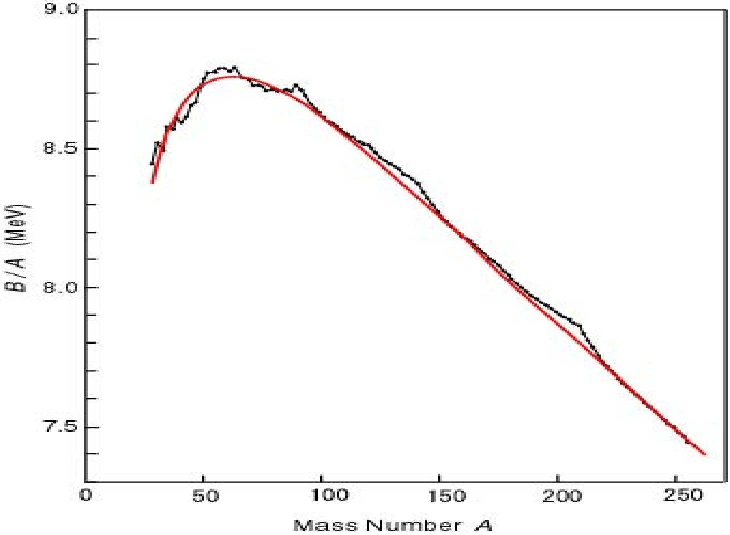
- Liquid drop model relies on empirical observation that nuclear radii are proportional to the cube root of the mass number A

$$R = r_0 A^{\frac{1}{3}} \implies S = 4\pi R^2 = 4\pi r_0^2 A^{\frac{2}{3}} \implies V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi r_0^3 A \quad (11)$$

- Liquid drop model predictions explain well binding energies per nucleon B/A observed for nuclei over a wide mass range.

$$B(Z,A)/A = a_V - a_S \frac{1}{A^{\frac{1}{3}}} - a_C \frac{Z(Z-1)}{A^{\frac{4}{3}}} - a_A \frac{(N-Z)^2}{A^2} + \delta(A,Z)/A \quad (12)$$

Liquid drop model's mass fit

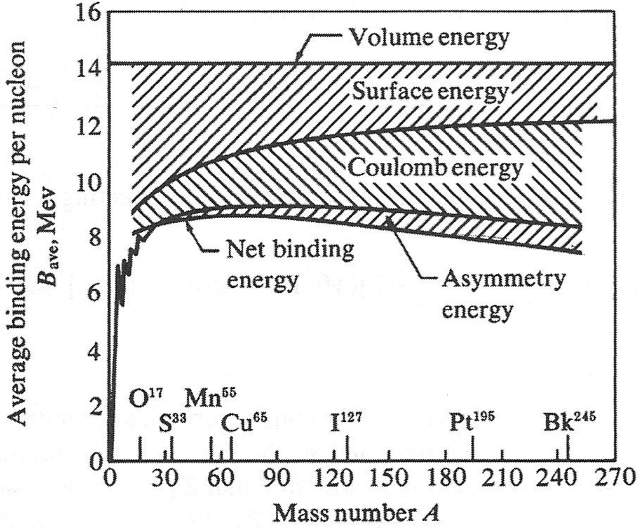


Liquid drop model's parameters

- The volume term coefficient $a_V = 15.56$ MeV.
- The surface term coefficient $a_S = 17.23$ MeV.
- The Coulomb term coefficient $a_C = 0.7$ MeV.
- The asymmetry term coefficient $a_V = 23.285$ MeV.
- The pairing term

$$\delta = \begin{cases} -\frac{11}{\sqrt{A}}\ [\text{MeV}] & \text{even-even nuclei} \\ 0\ [\text{MeV}] & \text{odd-even nuclei} \\ +\frac{11}{\sqrt{A}}\ [\text{MeV}] & \text{odd-odd nuclei} \end{cases} \tag{13}$$

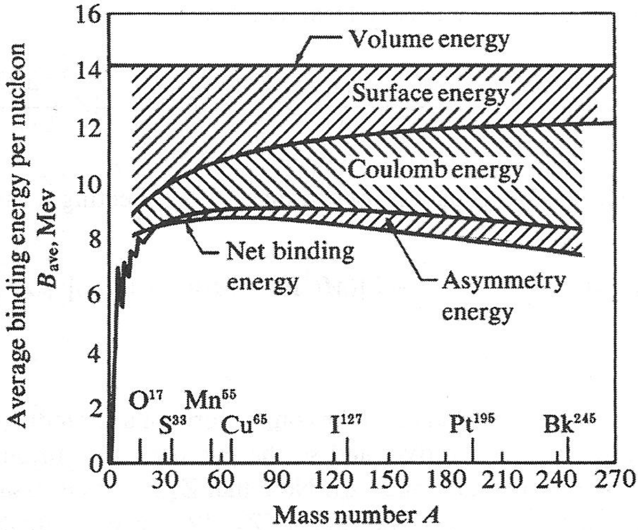
Liquid drop model's contribution



The volume term

- Volume energy results in binding. The volume coefficient is positive and large. The term is constant and dominates for isotopes with small and medium atomic number.
- Scaling of the binding energy with volume implies short range of effective nuclear interactions.
- Indeed, if the interaction is long range any nucleon will interact with all other nucleons. Then binding per nucleon would increase as number of nucleons increases. This is not observed.
- For short range interactions a nucleon interacts only with its immediate neighbours. This implies a constant binding energy per nucleon. This is indeed observed and referred to as saturation of nuclear interactions.
- Since the size of a nucleon is ~ 1 [fm] (10^{-15} [m]) the range of the effective nuclear interactions has to be comparable.

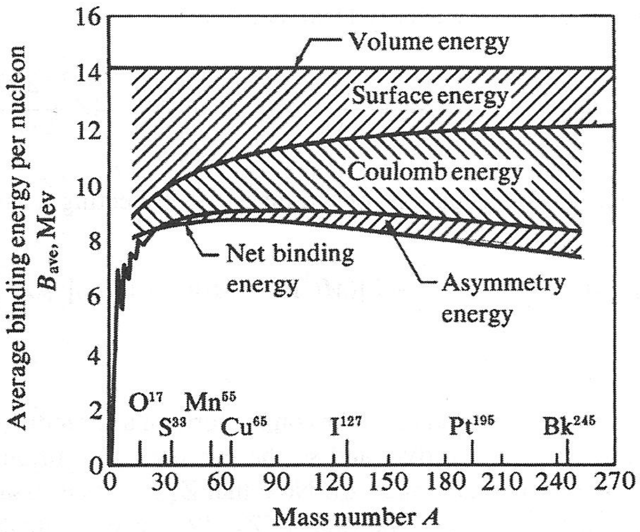
Liquid drop model's contribution



The surface term

- Surface energy reduces binding.
- Surface term accounts for the fact that a nucleon at the surface of a nucleus interacts with fewer other nucleons than one in the interior of the nucleus and hence its binding energy is less.
- The magnitude of the surface term coefficient is negative and comparable in magnitude to the volume term.
- Surface term plays a significant role in light nuclei which have large surface compared to their volume. It accounts for the drop in binding energy at light masses.

Liquid drop model's contribution



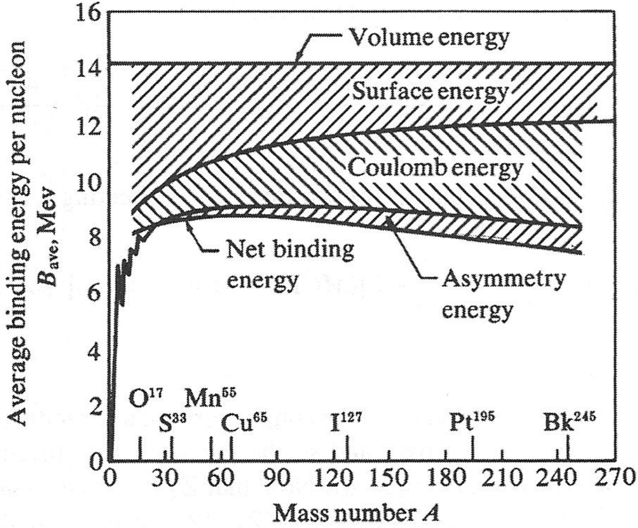
The Coulomb term

- Coulomb term accounts for electric repulsion between protons.
- An electrostatic energy contained within a uniformly-charged sphere of charge Ze and radius R can be calculated from the Coulomb law as

$$E_{Coulomb} = \frac{3}{5}\frac{1}{4\pi\varepsilon_0}\frac{Z^2e^2}{R} \tag{14}$$

- The Coulomb law implicates the mathematical form of the Coulomb term: direct proportionality to Z^2 and inverse proportionality to $A^{\frac{1}{3}}$ (radius).
- The coefficient of the Coulomb term is fitted, it is negative as the term reduces binding.
- Coulomb term play a significant role for isotopes with large Z (actinides). With increasing Z it becomes larger than the volume term and limits the existence of stable isotopes.

Liquid drop model's contribution



The asymmetry and pairing terms

- If it wasn't for the Coulomb energy, the most stable form of nuclear matter would have $N = Z = A/2$.
- This is a consequence of the charge-independence of nuclear interactions and a result of the Fermi gas model (which will be discussed later).
- Asymmetry energy is sometimes also called Pauli energy since there is a strong connection between the Fermi gas model and Pauli exclusion principle.
- Pairing term account for the odd-even effect shown in the separation energies of a neutron and a proton. It is a correction term that arises from the fact that an even number of same nucleons results in increased stability as compared to an odd number.

Line of stability

- Using the liquid drop model we can address a question of the most stable isotope $Z_A$for a given number of nucleons A.
- The most stable isotope will have the smallest mass.
- To find out Z_A we can calculate the masses as a function of Z and A using binding energies from the liquid drop model.
- Then for a fixed A we can search for a minimum mass using differential calculus (searching for a minimum as a function of Z by setting

$$\left.\frac{\partial M(Z,A)}{\partial Z}\right|_{A=\text{const.}} = 0. \tag{15}$$

Mass as a function of A and Z

- Using $N = A - Z$ in the asymmetry term mass as a function of A and Z is given by:

$$
\begin{aligned}
M(Z,A) &= ZM(^1H) + (A-Z)M(n) - \frac{B(Z,A)}{c^2} \\
B(Z,A) &= a_V A - a_S A^{\frac{2}{3}} - a_C \frac{Z^2}{A^{\frac{1}{3}}} - a_A \frac{(A-2Z)^2}{A} + \delta(A,Z).
\end{aligned}
$$

- Note that

$$(A-2Z)^2 = A^2 - 4AZ + 4Z^2 \tag{16}$$

Mass as a function of A and Z

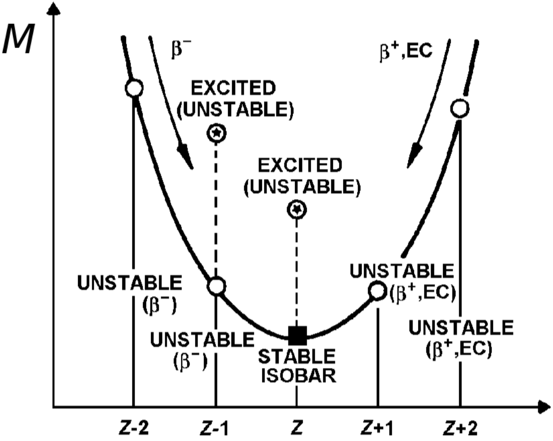
- Expanding the square asymmetry term and grouping terms proportional to Z^2, Z, and independent of Z yields

$$
\begin{aligned}
M(Z,A) &= Z^2\left[\frac{4a_A}{A}+\frac{a_C}{A^{\frac{1}{3}}}\right]+ \\
&+ Z\left[M(^1H)-M(n)-4a_A\right]+ \\
&+ A\left[M(n)-a_V+\frac{a_S}{A^{\frac{1}{3}}}+a_A\right]+ \\
&+ \delta(A,Z) \qquad (17)
\end{aligned}
$$

- Note that mass dependence on Z at the fixed A is parabolic.

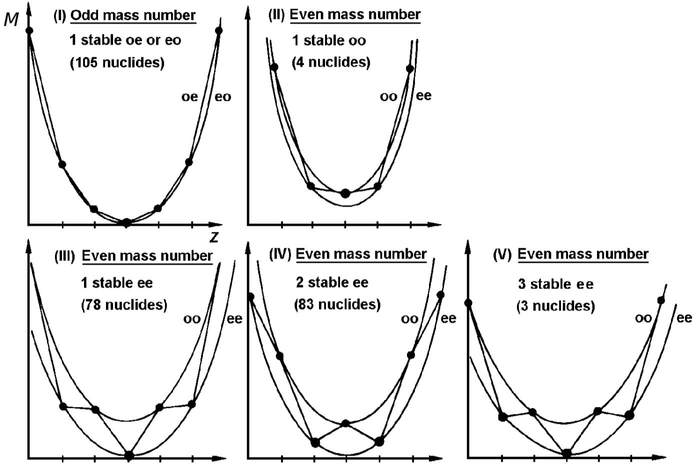
Odd-mass nuclei

- In odd-mass nuclei the pairing term is zero. The isobaric (fixed A) mass as a function of Z is shown on the graph



Even-mass nuclei

- In even-mass nuclei the pairing term is different for the even-even and odd-odd case. The isobaric mass parabolas are:



The minimum of the mass parabola

- The condition

$$\left.\frac{\partial M(Z,A)}{\partial Z}\right|_{A=\text{const.}} = 0. \tag{18}$$

implicates

$$\begin{aligned} 0 &= 2Z_A\left[\frac{4a_A}{A}+\frac{a_C}{A^{\frac{1}{3}}}\right]+ \\ &+ \left[M(^1H)-M(n)-4Za_A\right] \end{aligned} \tag{19}$$

- The (non-integer) solution is

$$\frac{Z_A}{A} = \frac{1}{2}\frac{4a_A+M(n)-M(^1H)}{4a_A+a_CA^{\frac{2}{3}}} = \frac{1}{2}\frac{81}{80+0.6A^{\frac{2}{3}}} \tag{20}$$

- The most stable isotope have integer Z near the Z_A value.

The line of stability

- The condition

$$\frac{Z_A}{A} = \frac{81}{80 + 0.6A^{\frac{2}{3}}} \tag{21}$$

implicates that $Z/A = 0.5$ for small A and $Z/A < 0.5$ for large A.

- This is indeed observed and result from the balance between the Coulomb repulsion between protons in a nucleus which increases with Z and the volume binding which increases with A.
- In heavy nuclei excess of neutrons provides nuclear binding compensating Coulomb repulsion between protons.
- However, at some large Z the Coulomb repulsion overcomes the volume binding and stable isotopes cease to exists.

Line of stability