Scattering Processes

General Consideration

Kinematics of electron scattering

Fermi Golden Rule

Rutherford scattering cross section

The form factor

Mott scattering

Nuclear charge distributions and radii

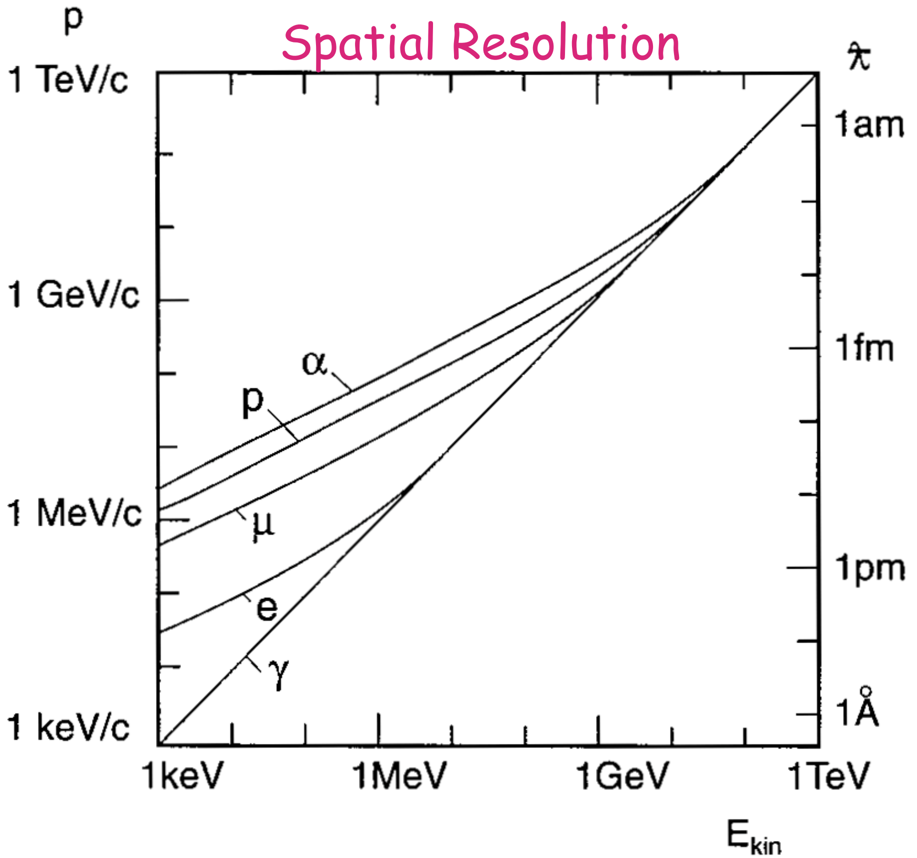
Momentum, kinetic energy and reduced wavelength

$$\frac{\lambda}{2\pi} = \frac{\hbar}{|p|} = \frac{\hbar c}{\sqrt{2mc^2 E_{kin} + E_{kin}}} \approx \begin{cases} \frac{\hbar}{\sqrt{2mE_{kin}}}, & E_{kin} << mc^2 \\ \frac{\hbar c}{E_{kin}} \approx \frac{\hbar c}{E}, & E_{kin} >> mc^2 \end{cases}$$

$$\frac{\lambda}{2\pi} \leq \Delta x \Leftrightarrow |p| \geq \frac{\hbar}{\Delta x}$$

$$|p|c \geq \frac{\hbar c}{\Delta x} \approx \frac{200\,MeV\ \ fm}{\Delta x}$$

That means that I need e^- with 100 MeV/c to investigate the nucleus size (fm)

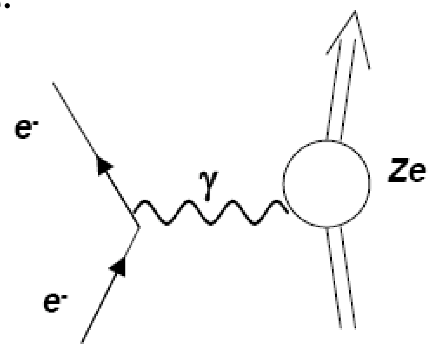


Elastic Electron Scattering

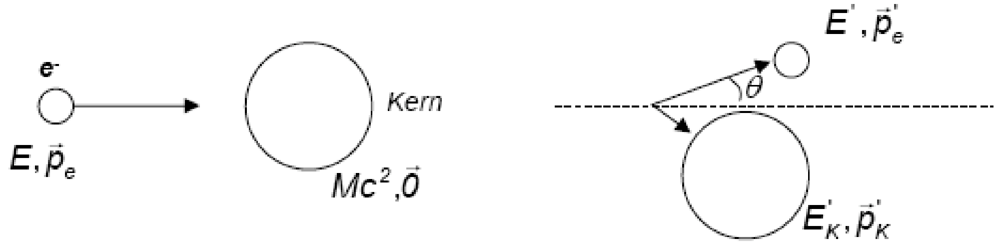
The electron elastic scattering serves to investigate the electric charge distribution of nucleus and nucleons.

The elastic scattering is carried out by a virtual Photon.

Electrons were chosen because they are pointlike and interact with the target via the electromagnetic interaction



Kinematic of the Electron Scattering



Taking the 4-vectors:

K=N

$$P_e + P_N = P_e' + P_N'$$
$$\rightarrow P_e^2 + 2P_eP_N + P_N^2 = P_e'^2 + 2P_e'P_N' + P_N'^2$$

The elastic scattering does not modify the invariant mass of the scattering particles:

$$P_e^2 = P_e'^2 = m_e^2c^2, \; P_N^2 = P_N'^2 = M^2c^2$$
$$\rightarrow P_eP_N = P_e'P_N'$$

Since experimentally the scattered nucleus is not identified one assigns:

$$P_N' = P_e + P_N - P_e'$$

$$P_e P_N = P_e^{'}(P_e + P_N - P_e^{'}) = P_e^{'} P_e + P_e^{'} P_N - m_e^2 c^2$$

The target stays still in the lab system before the impact

$$P_N = (E_N / c, \vec{p}_N) = (Mc, 0)$$

$$P_e = (E / c, \vec{p}_e), \quad P_e^{'} = (E^{'} / c, \vec{p}'_e), \quad P_N^{'} = (E_N^{'} / c, \vec{p}'_N)$$

$$EM = \frac{EE^{'}}{c^2} - \vec{p}_e \vec{p}_e^{'} + E^{'} M - m_e^2 c^2 \rightarrow EMc^2 = EE^{'} - \vec{p}_e \vec{p}_e^{'} c^2 + E^{'} Mc^2 - m_e^2 c^4$$

For relativistic electrons ($E \gg m_e c^2$) the last term can be neglected

$$E \approx |\vec{p}_e| c \rightarrow EMc^2 = EE^{'}(1 - \cos\theta) + E^{'} Mc^2$$

$$E^{'} = \frac{E}{1 + E / Mc^2 (1 - \cos\theta)}$$

Where θ is the scattering angle.

Which informations about the target particle can be extracted from the measurement of the scattering angle?

For this purpose one has to compute the differential cross-section $d\sigma/d\Omega$

Energy of electrons scattered by a nucleus, as a function of scattering angle

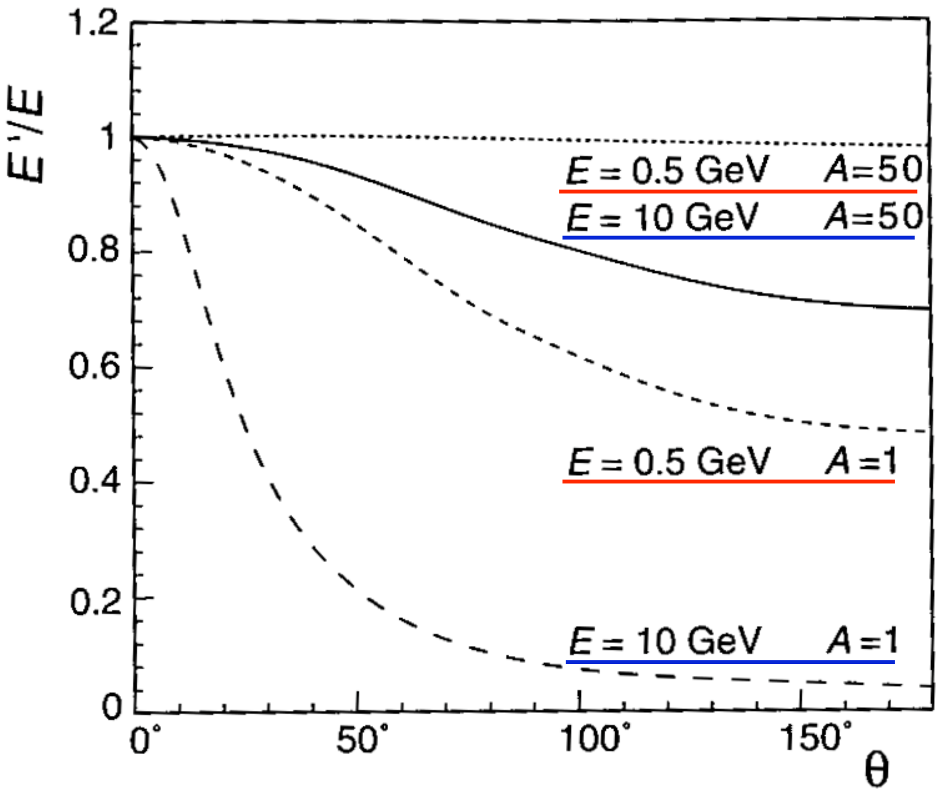


Abb. 5.2. Winkelabhängigkeit der auf die Strahlenergie normierten Elektronstreuenergie E'/E bei elastischer Elektron-Kern-Streuung. Die Kurven zeigen diesen Zusammenhang für zwei verschiedene Strahlenergien (0.5 GeV und 10 GeV) und zwei unterschiedlich schwere Kerne ($A = 1$ und $A = 50$).

(Povh, ..., T&K)

Fermi Golden Rule

The probability that a reaction between an incoming beam particle and a target takes place depends on the Interaction Potential (H_{int}) between the two particles. Given the two wave functions corresponding to the incoming (ψ_I) and outgoing electron (ψ_F), one can calculate the Matrix Element M_{FI} as:

$$M_{FI} = \langle \psi_F | H_{\text{int}} | \psi_I \rangle = \int \psi_F^* H_{\text{int}} \psi_I \, dV$$

Where H_{int} is the hamiltonian operator of the corresponding interaction. Furthermore the reaction rate depends on the possible number of final states of the outgoing particle. To calculate this number one has to consider that each particle occupies a volume in the phase space equal to:

$$h^3 = (2\pi\hbar)^3$$

Together with the Volume V we consider the Momentum interval ‚p+dp' and its correlated volume in the Momentum space:

$$V_{momentum} = 4\pi \cdot p^2 dp$$

We can calculate the total number of possible states as:

$$dn(p') = \frac{V \cdot 4\pi \cdot p'^2 dp'}{(2\pi\hbar)^3}$$

Total available volume

$$\frac{dE'}{dp'} = \frac{d(1/2mv'^2)}{d(mv')} = v' \Rightarrow dE' = v' dp'$$

One can write the density of the final state ρ(E') in the energy interval dE':

$$\rho(E') = \frac{dn(E')}{dE'} = \frac{V \cdot 4\pi \cdot p'^2}{v'(2\pi\hbar)^3}$$

M_{FI} and ρ(E) are the main constituents of the Fermi Golden Rule that defines the reaction rate W per beam and target particle like:

$$W = \frac{2\pi}{\hbar} \cdot |M_{FI}|^2 \cdot \rho(E')$$

One can write the reaction Rate (R) per Beam (N_b) and target particle (N_a):

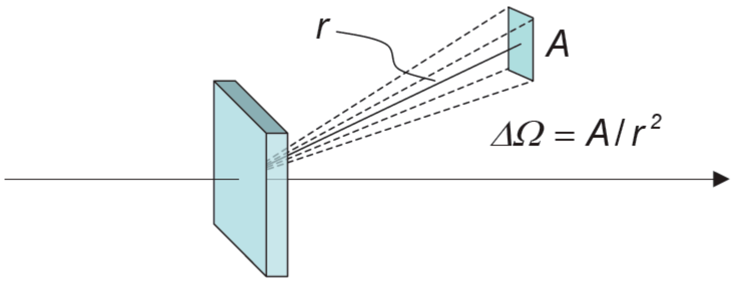
$$W = \frac{R}{N_a N_b} = \frac{I \cdot n_b \cdot \sigma}{N_a N_b} = \frac{I \cdot \sigma}{N_a A}, \quad n_b = N_b / A$$

$$I = \frac{N_a}{dt} \frac{dx}{dx} \frac{A}{A} = v_a \rho_a A, \qquad \rho_a = \frac{N_a}{V}$$

$$W = \frac{v_a \cdot \sigma}{V}$$

Where I is the number of incoming particle per time unit

Using the Fermi golden Rule:



$$\sigma = \frac{2\pi}{\hbar \cdot v_a} \cdot |M_{FI}|^2 \cdot \rho(E') \cdot V$$

If we consider relativistic electrons, we have: E'=p'c and v_a=v'=c

We can define the phase factor ρ(E')

$$\rho(E') = \frac{dn}{dE'} = \frac{V \cdot 4\pi \cdot E'^2}{c^3 (2\pi\hbar)^3}$$

$$\sigma = |M_{FI}|^2 \cdot \frac{V^2 \cdot 4\pi \cdot E'^2}{(2\pi)^2 (\hbar c)^4}$$

If we considering the scattering in the solid angle dΩ

$$d\sigma = |M_{fi}|^2 \cdot \frac{V^2 \cdot E'^2}{(2\pi)^2 (\hbar c)^4} d\Omega \quad \text{bzw.} \quad \frac{d\sigma}{d\Omega} = |M_{fi}|^2 \cdot \frac{V^2 \cdot E'^2}{(2\pi)^2 (\hbar c)^4}$$

In order to calculate the Matrixelement M_{FI} we have to define an incoming and outcomina wave

$$\psi_i = \frac{1}{\sqrt{V}} e^{\frac{ipx}{\hbar}}, \quad \psi_f = \frac{1}{\sqrt{V}} e^{\frac{ip'x}{\hbar}}$$

Born Approximation

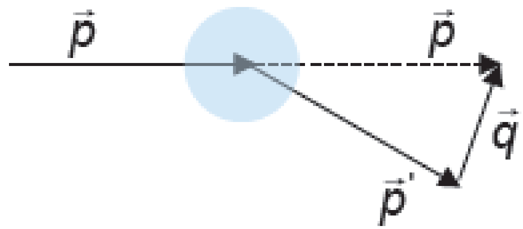
If we consider a Charge e inside an electric potential $\phi(x)$, the Interaction operator H_{int} will be:

$$H_{\text{int}} = e \cdot \phi(x)$$

$$\Rightarrow M_{fi} = \langle \psi_f \,|\, H_{\text{int}} \,|\, \psi_i \rangle = \frac{e}{V} \cdot \int e^{-\frac{i\vec{p}'\vec{x}}{\hbar}} \phi(x)\, e^{\frac{i\vec{p}\vec{x}}{\hbar}} d^3x$$

<u>Momentum transfer</u>

$$\vec{q} = \vec{p} - \vec{p}'$$



$$M_{fi} = \langle \psi_f \,|\, H_{\text{int}} \,|\, \psi_i \rangle = \frac{e}{V} \cdot \int \phi(x)\, e^{\frac{i\vec{q}\vec{x}}{\hbar}} d^3x$$

Using the Green Theorem for u and v being two scalar functions which go to 0 for large r:

$$\int (u\Delta v - v\Delta u) d^3r = 0 \Rightarrow \qquad \int u \cdot \Delta v \, d^3x = \int v \cdot \Delta u \, d^3x$$

In our case one can write:

$$e^{\frac{i\vec{q}\vec{x}}{\hbar}} = -\frac{\hbar^2}{|\vec{q}|^2} \cdot \Delta e^{\frac{i\vec{q}\vec{x}}{\hbar}}$$

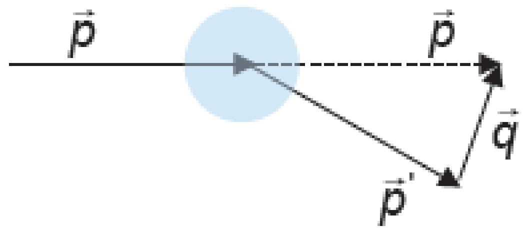
If we consider a Charge e inside an electric potential $\phi(x)$, the Interaction operator H_{int} will be:

$$H_{\text{int}} = e \cdot \phi(x)$$

$$\Rightarrow M_{fi} = \langle \psi_f \mid H_{\text{int}} \mid \psi_i \rangle = \frac{e}{V} \cdot \int e^{-\frac{i\vec{p}'\vec{x}}{\hbar}} \phi(x)\, e^{\frac{i\vec{p}\vec{x}}{\hbar}} d^3x$$

<u>Momentum transfer</u>

$$\vec{q} = \vec{p} - \vec{p}'$$



$$M_{fi} = \langle \psi_f \mid H_{\text{int}} \mid \psi_i \rangle = \frac{e}{V} \cdot \int \phi(x)\, e^{\frac{i\vec{q}\vec{x}}{\hbar}} d^3x$$

Using the Green Theorem for u and v being two scalar functions which go to 0 for large r:

$$\int (u\Delta v - v\Delta u) d^3 r = 0 \Rightarrow$$

$$\iiint_V (P\Delta Q - Q\Delta P) d^3 r = \iint_S \left(P\frac{\partial Q}{\partial n} - Q\frac{\partial P}{\partial n} \right) d^2 r$$

$$\Delta Q = \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2}, \quad \frac{\partial Q}{\partial n} = gradient$$

In our case one can write:

$$e^{\frac{i\vec{q}\vec{x}}{\hbar}} = -\frac{\hbar^2}{|\vec{q}|^2} \cdot \Delta e^{\frac{i\vec{q}\vec{x}}{\hbar}}$$

$$M_{fi} = \langle \psi_f \mid H_{\text{int}} \mid \psi_i \rangle = -\frac{e\hbar^2}{V|\vec{q}|^2} \cdot \int \Delta\phi(x)\, e^{\frac{i\vec{q}\vec{x}}{\hbar}}\, d^3x$$

Poisson equation: $\nabla E = \nabla(-grad\ \phi) = -\Delta\phi = \frac{\rho(x)}{\varepsilon_0}$ $\boxed{\nabla^2 = \Delta}$

If we consider a normalized charge distribution f(x) we can write

$$\rho(x) = Ze \cdot f(x) \qquad \int f(x) d^3x = 1 \qquad \Delta\phi = -\frac{\rho(x)}{\varepsilon_0} = -\frac{Ze}{\varepsilon_0} f(x)$$

$$M_{fi} = \langle \psi_f \mid H_{\text{int}} \mid \psi_i \rangle = -\frac{Z \cdot 4\pi \cdot \alpha \cdot \hbar^3 c}{V \cdot |\vec{q}|^2} \cdot \underbrace{\int f(x)\, e^{\frac{i\vec{q}\vec{x}}{\hbar}}\, d^3x}_{F(\vec{q})}, \qquad \text{with} \quad \alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$$

F(q) is the Fourier Transformation of the Charge distribution:

Form Factor

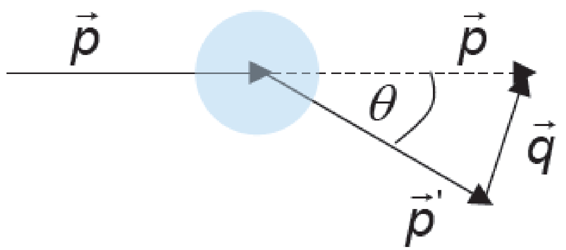
$$\frac{d\sigma}{d\Omega} = \frac{V^2 \cdot E'^2}{(2\pi)^2 (\hbar c)^4} \cdot |M_{fi}|^2 = \frac{4 \cdot Z^2 \cdot \alpha^2 \cdot (\hbar c)^2 E'^2}{|\vec{q}c|^4} \cdot |F(\vec{q})|^2$$

Rutherford Scattering

- Pointlike charge that scatter on pointlike target (No inner structure->f(x)=δ(x))
- The target is heavy and hence the recoil energy can be neglected
- Spin 0 particles

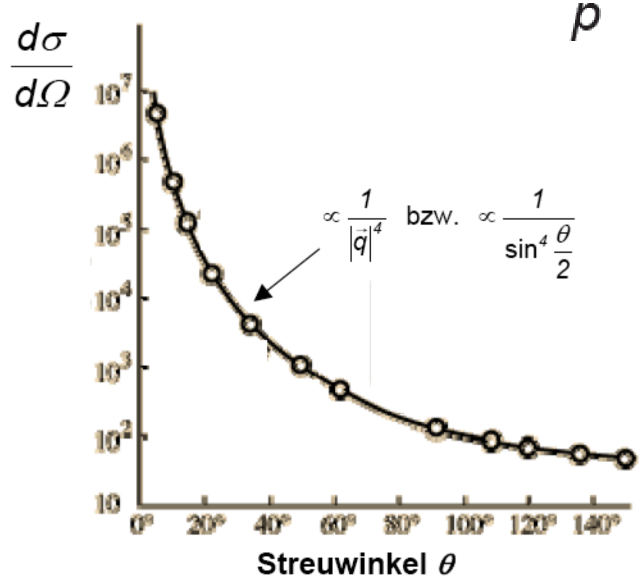
$$\frac{d\sigma}{d\Omega} = \frac{4 \cdot Z^2 \cdot \alpha^2 \cdot (\hbar c)^2 E'^2}{|\vec{q}c|^4} \cdot 1 = \left(\frac{d\sigma}{d\Omega}\right)_{Rutherford}$$

$$|\vec{p}'| \cong |\vec{p}| \qquad E \cong |\vec{p}| \cdot c$$



$$|\vec{q}| = 2 \cdot |\vec{p}| \sin\frac{\theta}{2} = 2 \cdot \frac{E}{c} \sin\frac{\theta}{2}$$

$$\rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} = \frac{Z^2 \cdot \alpha^2 \cdot (\hbar c)^2}{4E^2 \sin^4\frac{\theta}{2}}$$



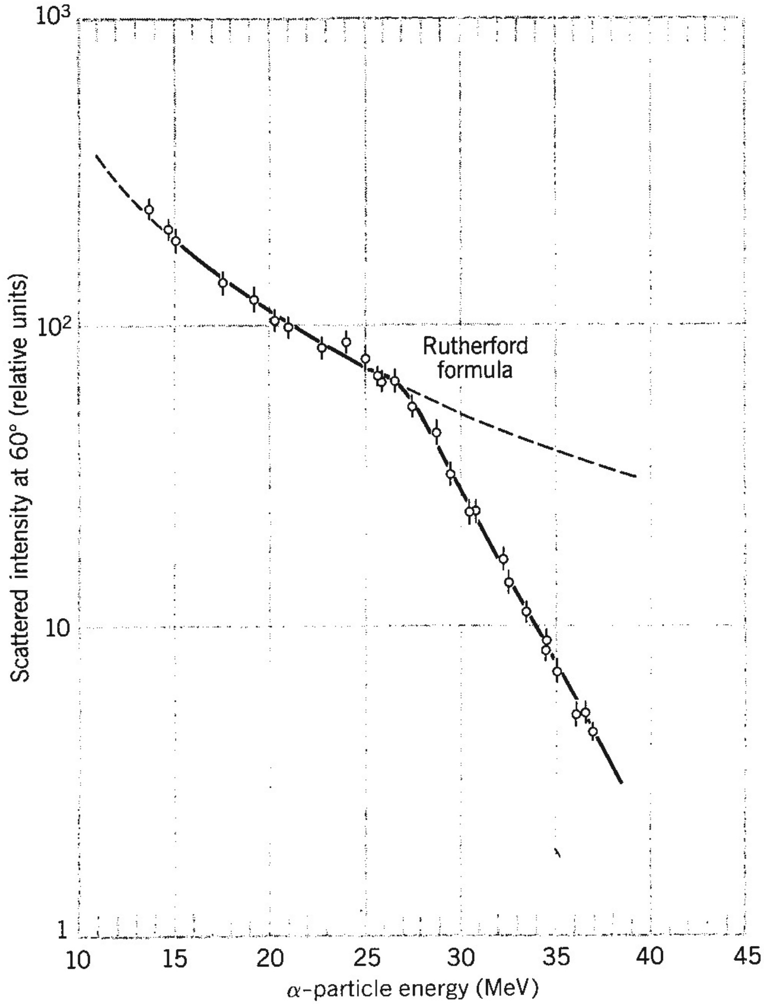
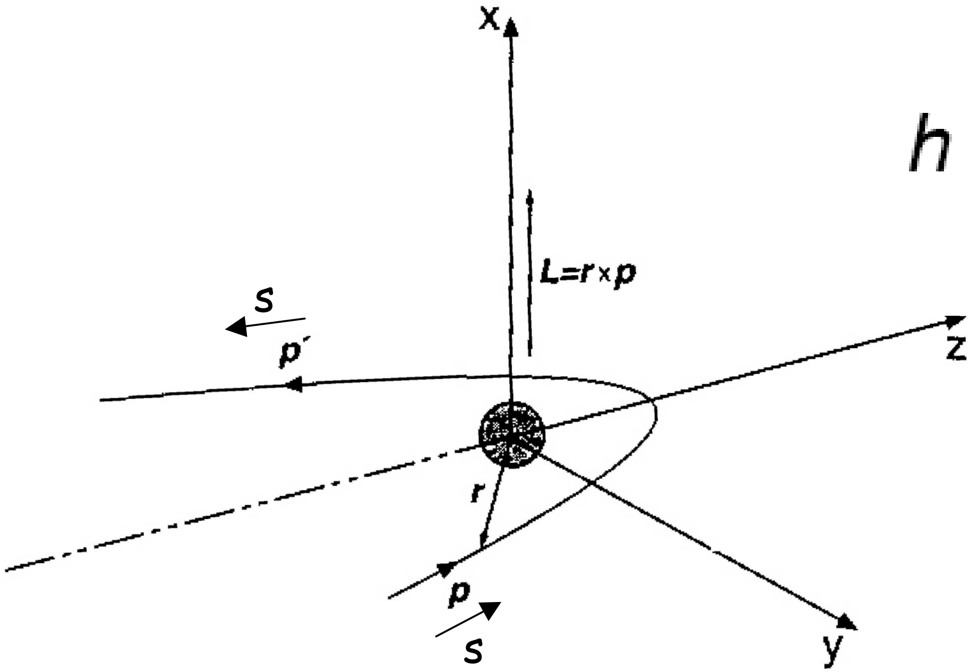


Figure 3.11 The breakdown of the Rutherford scattering formula. When the incident α particle gets close enough to the target Pb nucleus so that they can interact through the nuclear force (in addition to the Coulomb force that acts when they are far apart) the Rutherford formula no longer holds. The point at which this breakdown occurs gives a measure of the size of the nucleus. Adapted from a review of α particle scattering by R. M. Eisberg and C. E. Porter, *Rev. Mod. Phys.* **33**, 190 (1961).

Helicity suppression of electron backscattering



$$h = \frac{\vec{s} \cdot \vec{p}}{|\vec{s}| \cdot |\vec{p}|}$$

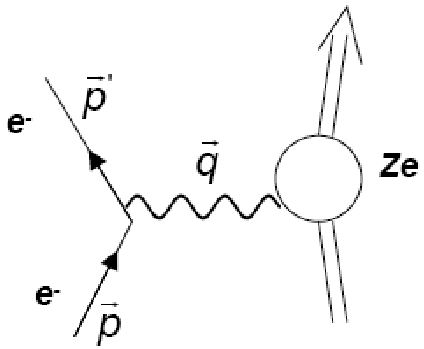
Fig. 5.3. Helicity, $h = \boldsymbol{s} \cdot \boldsymbol{p}/(|\boldsymbol{s}| \cdot |\boldsymbol{p}|)$, is conserved in the $\beta \to 1$ limit. This means that the spin projection on the z-axis would have to change its sign in scattering through 180°. This is impossible if the target is spinless, because of conservation of angular momentum.

$$\left(\frac{d\sigma}{d\Omega}\right)^{*}_{Mott} = \left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} \cdot \left(1 - \beta^2 \sin^2 \frac{\theta}{2}\right)$$

!!!The recoil of the target nucleus is neglected!!

Suppresed at 180°

Form Factor of the Nuclei



Electron scattering with a nucleus with a charge distribution. The momentum is carried by the photon which wave length determines the accuracy of the measurement

$$\lambda\!\!\!^{-} = \frac{\hbar}{|\vec{q}|} = \frac{\hbar}{|\vec{p}| \cdot 2\sin\frac{\theta}{2}}$$

small q

Photon couples to the total charge of the nucleus

big q

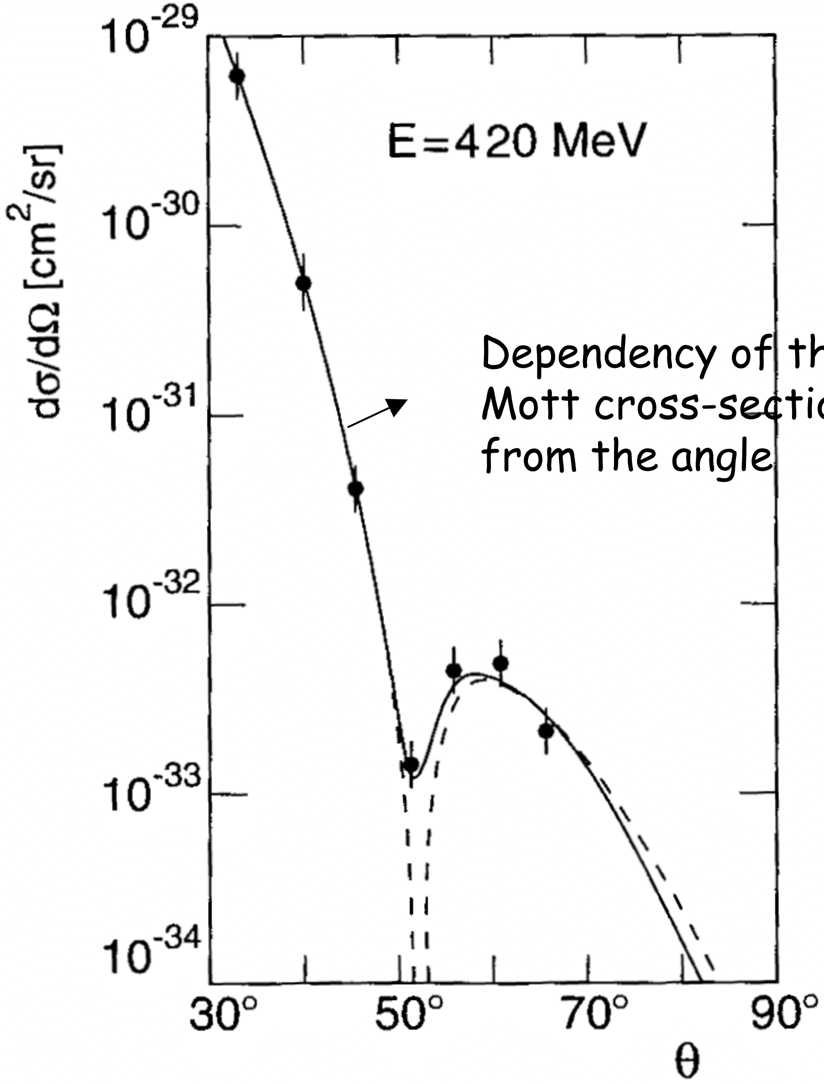
Photon couples to only some fraction of the charge

$$\left(\frac{d\sigma}{d\Omega}\right)_{exp} = \left(\frac{d\sigma}{d\Omega}\right)^{*}_{Mott} \cdot \left|F(q^2)\right|^2$$

For pointlike charge

For symmetric charge distributions the Form Factor depends only on the absolute value of the momentum q.

Differential Cross section for electron scattering from ^{12}C:



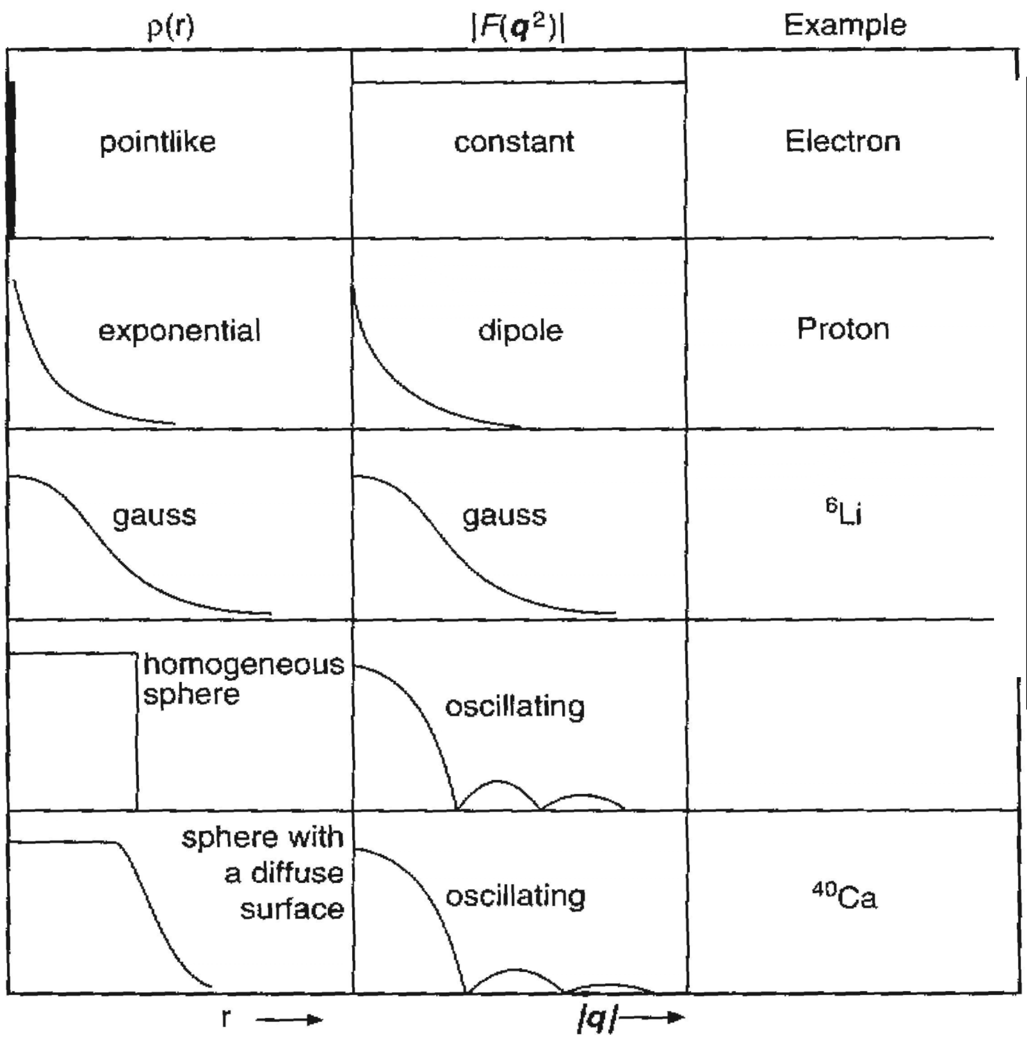
Dependency of the Mott cross-section from the angle

First experiments in the 50s at SLAC

The charachteristic interference picture shows the finite dimension of the nucleus, but what about the form?

(Povh, ..., P & N)

Form factors (Povh..., Particles & nuclei)



Charge Distribution	Form Factor F(q²)
$\delta(r)/4\pi$	1
$(a^3/8\pi)\cdot\exp(-ar)$	$(1+q^2/a^2\hbar^2)^{-2}$
$\left(\frac{a^2}{2\pi}\right)^{\frac{3}{2}}\exp\left(-\frac{a^2r^2}{2}\right)$	$\exp\left(-\frac{q^2}{2a^2\hbar^2}\right)$
$\begin{cases}\frac{3}{4\pi}\cdot R^3 & \text{für } r\le R\\ 0 & \text{für } r>R\end{cases}$	$3\alpha^{-3}(\sin\alpha-\alpha\cos\alpha)$ mit $\alpha=\frac{\lvert\vec{q}\rvert R}{\hbar}$

For the Homogeneus sphere the first minimum ist at:

$$\alpha=\frac{\lvert\vec{q}\rvert\cdot R}{\hbar}\approx 4{,}5$$

One can determine the size of the nucleus R.

Nuclear charge density ↔ nucleon (number)density

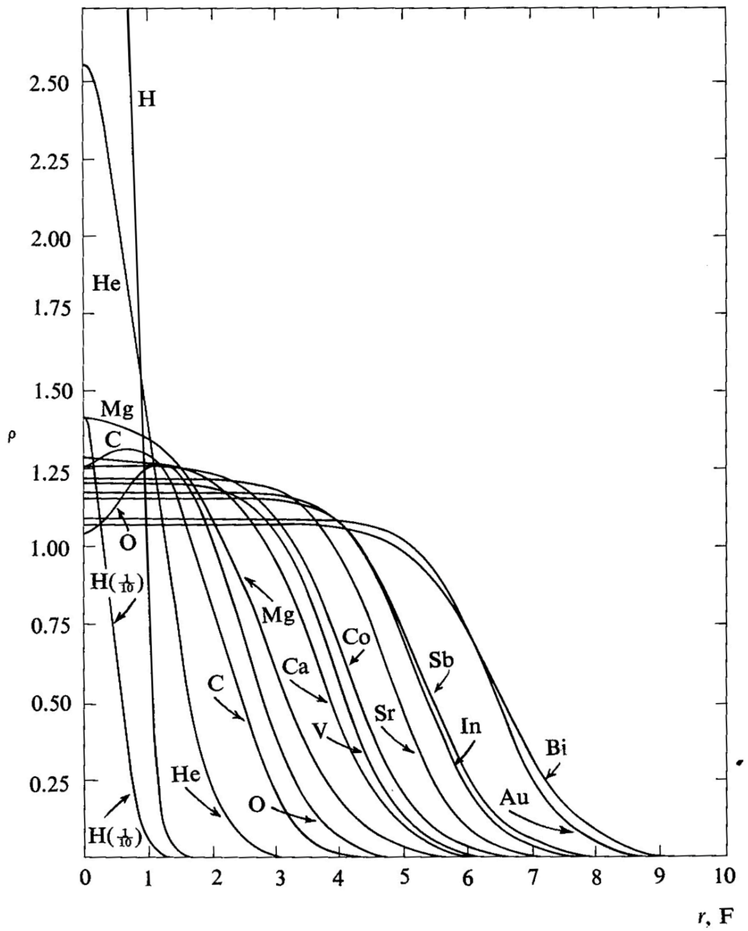


Figure 6-9 Nuclear charge density as a function of distance from the center of the nucleus foun by electron scattering methods. Ordinates unit: 10^{19} coulomb cm^{-3}. [R. Hofstadter, *Ann. Rev. Nuc Sci.*, **7**, 231 (1957).]

(Segrè, Nuclei & particles)

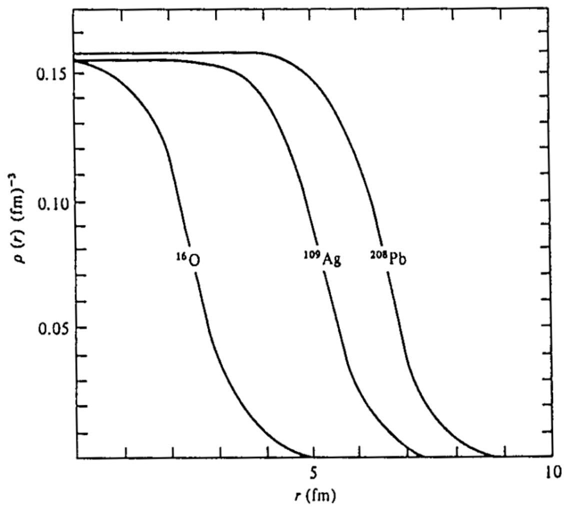
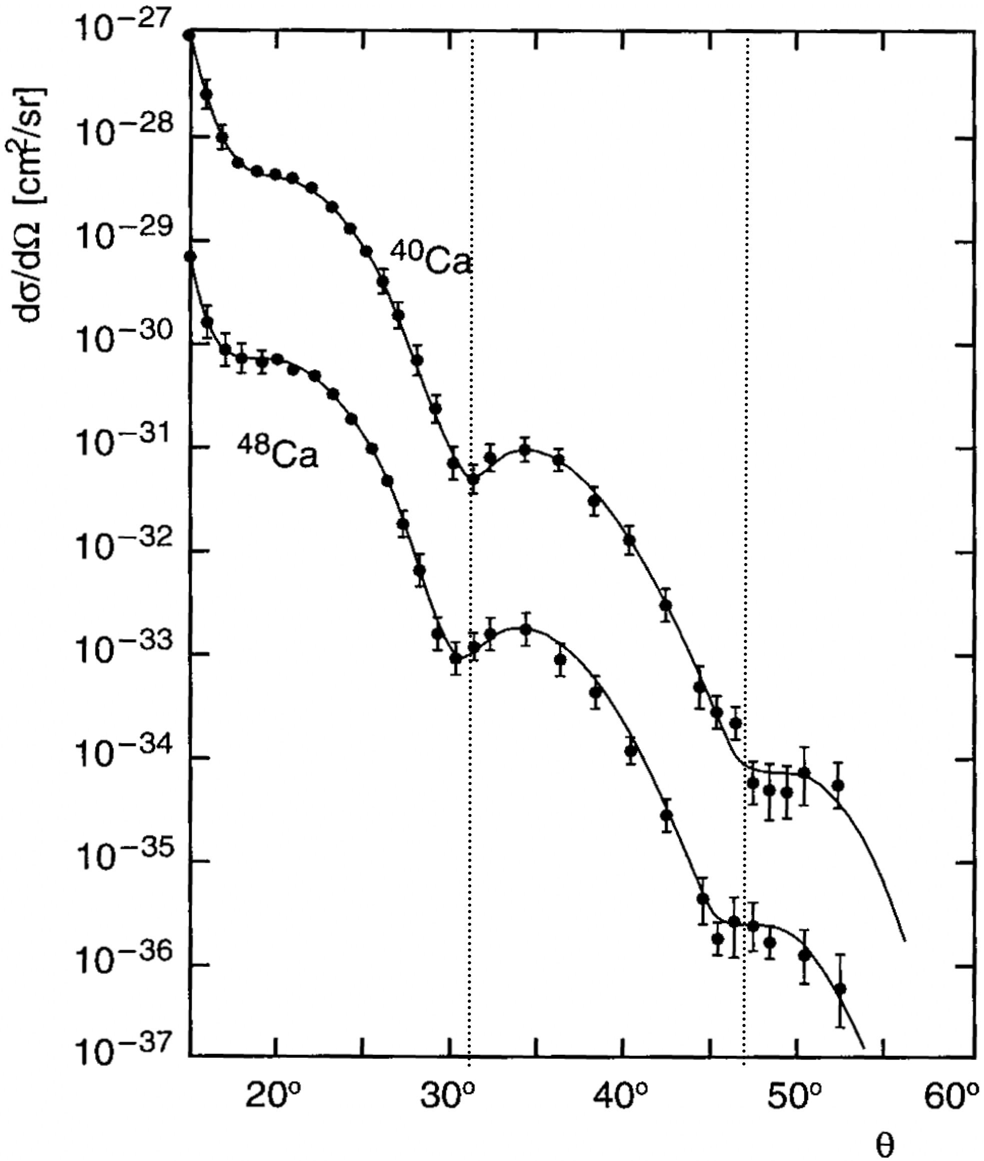


Abb. 1.12: Die Dichte der Nukleonen im Kern [Nach W.N. Cottingham et al. [9]]

(Bopp, Kerne...)

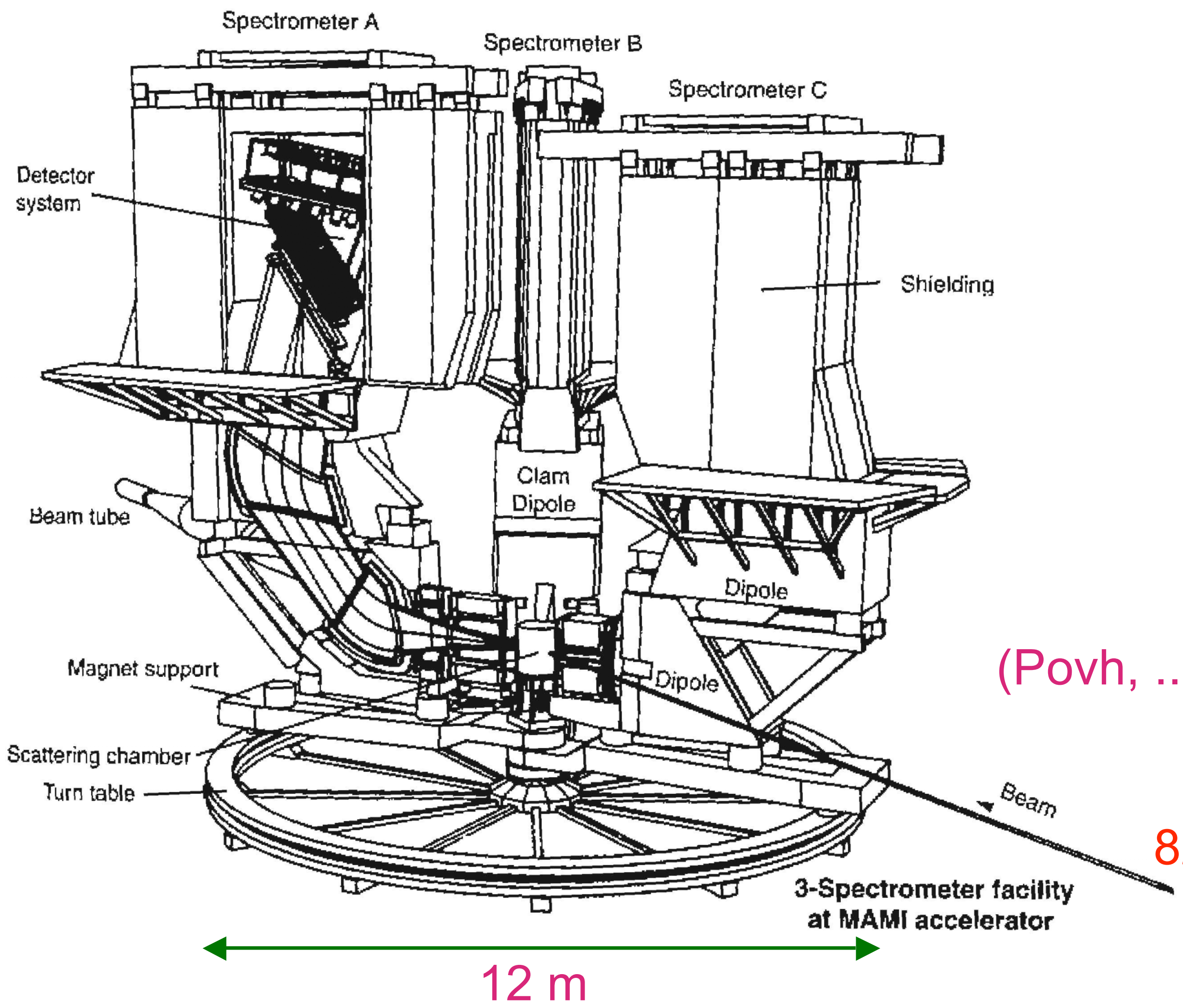
Differential cross section for
electron scattering from ^{40}Ca and ^{48}Ca:



The Form Factor of Isotopes is sligthly different since the neutron rich nuclei present the minima for smaller q values. Indeed for equal number of protons the nuclear volume becomes bigger.

(Povh, ..., P & N)

Electron spectrometer MAMI - B
at the Mainz microtron



(Povh, ..., P & N)

820 MeV

Mean Squared Charge Radius

Informations about the Nuclear radius can be extracted looking at the behaviour of $F(q^2)$ for small q..

$$if\ \frac{|q|\cdot R}{\hbar} \ll 1$$

$$F(q^2) = \int f(r) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i|q||r|\cos\vartheta}{\hbar} \right)^n d^3r$$

$$= \int_0^{\infty} \int_{-1}^{+1} \int_0^{2\pi} f(r) \left[1 - \frac{1}{2} \left(\frac{|q|\cdot R}{\hbar} \right)^2 \cos^2\vartheta + .. \right] d\phi\, d\cos\vartheta\, r^2 dr$$

$$= 4\pi \int_0^{\infty} f(r) r^2 dr - \frac{1}{6} \frac{q^2}{\hbar^2} 4\pi \int_0^{\infty} f(r) r^4 dr + ...$$

Since the charge distribution function $f(q^2)$ is normalized, one can define the mean square charge radius as

$$< r^2 > = 4\pi \int_0^{\infty} r^2 \cdot f(r) r^2 dr$$

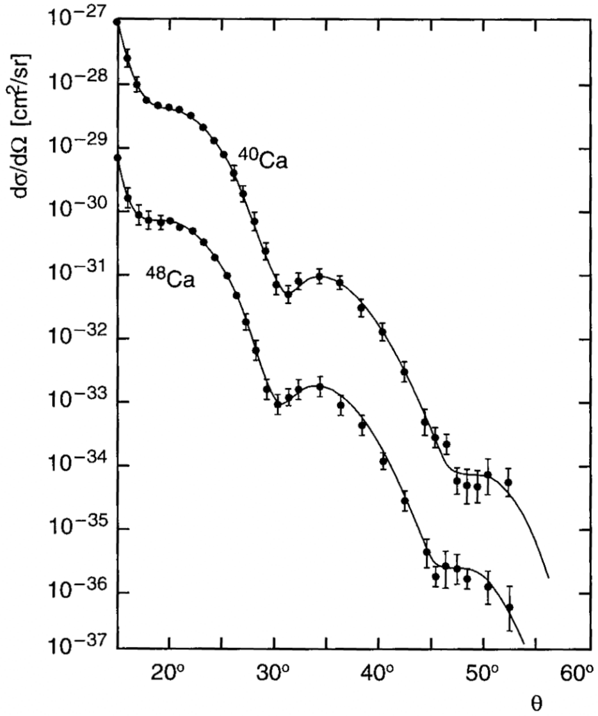
$$F(q^2) = 1 - \frac{1}{6} \frac{q^2 < r^2 >}{\hbar^2} + ...$$

$$< r^2 > = -6\hbar^2 \left. \frac{dF(q^2)}{dq^2} \right|_{q^2=0}$$

500 MeV Electrons:

atomic nuclei

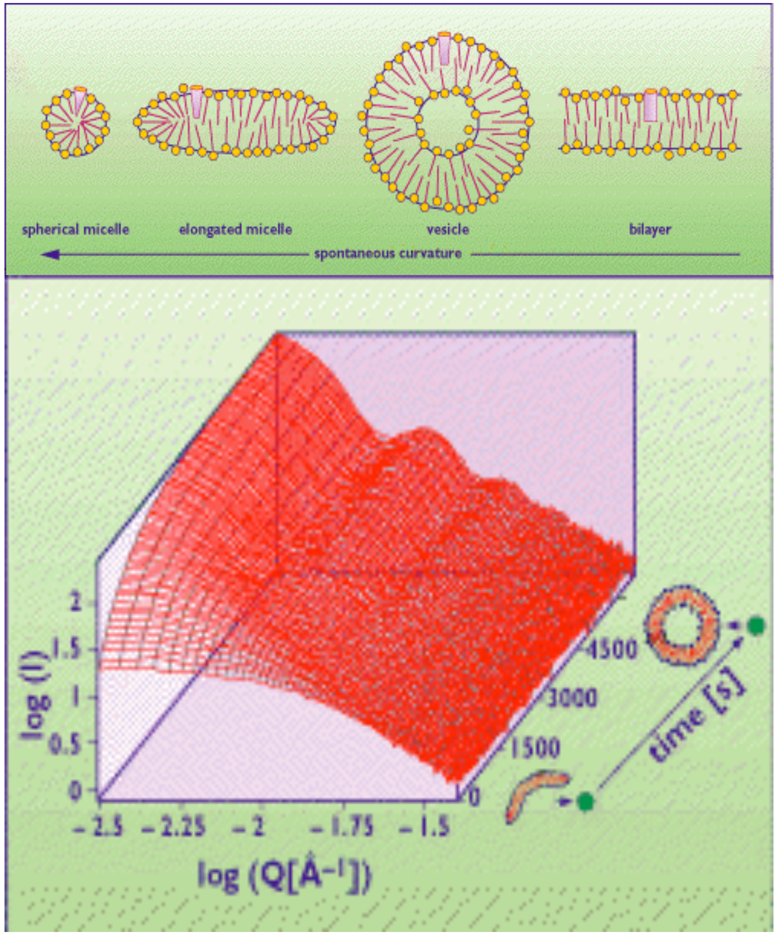
(a few 10^{-15} m)



10 meV neutrons:

macromolecules

(a few 10^{-9} m)



ILL annual report 1996 (soft matter)

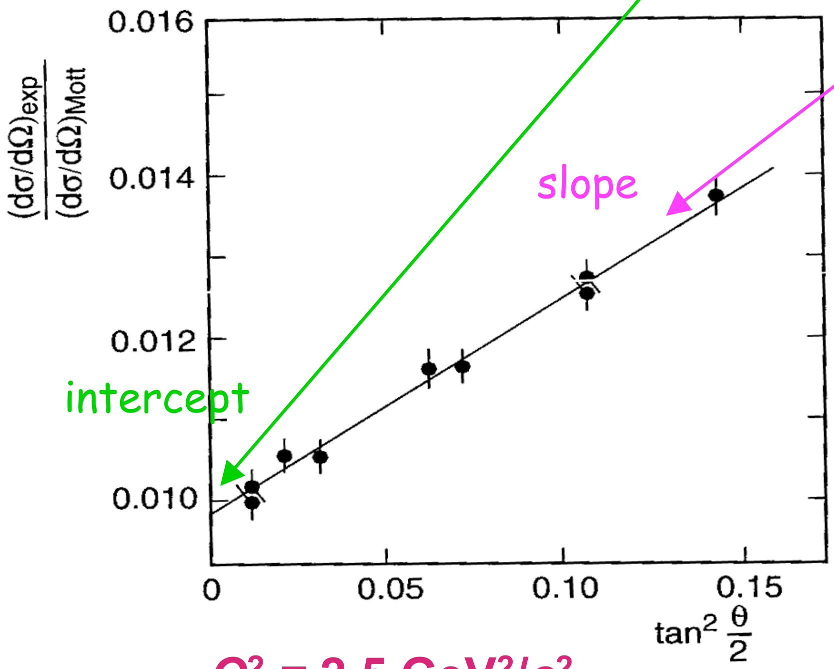
www.ill.fr

Rosenbluth plot

The e^- charge interacts also with the nuclear magnetic momentum $\mu = g\frac{e}{2M}\cdot\frac{\hbar}{2}$

$$\left(\frac{d\sigma}{d\Omega}\right)_{spin\,1/2} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left[1 + 2\tau\tan^2\frac{\vartheta}{2}\right], \quad \tau = \frac{Q^2}{4M^2c^2}$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left[\frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1+\tau} + 2\tau G_M^2(Q^2)\tan^2\frac{\vartheta}{2}\right]$$



$Q^2 = 2.5$ GeV$^2/c^2$

$G_E(Q^2)$ electric Formfactor

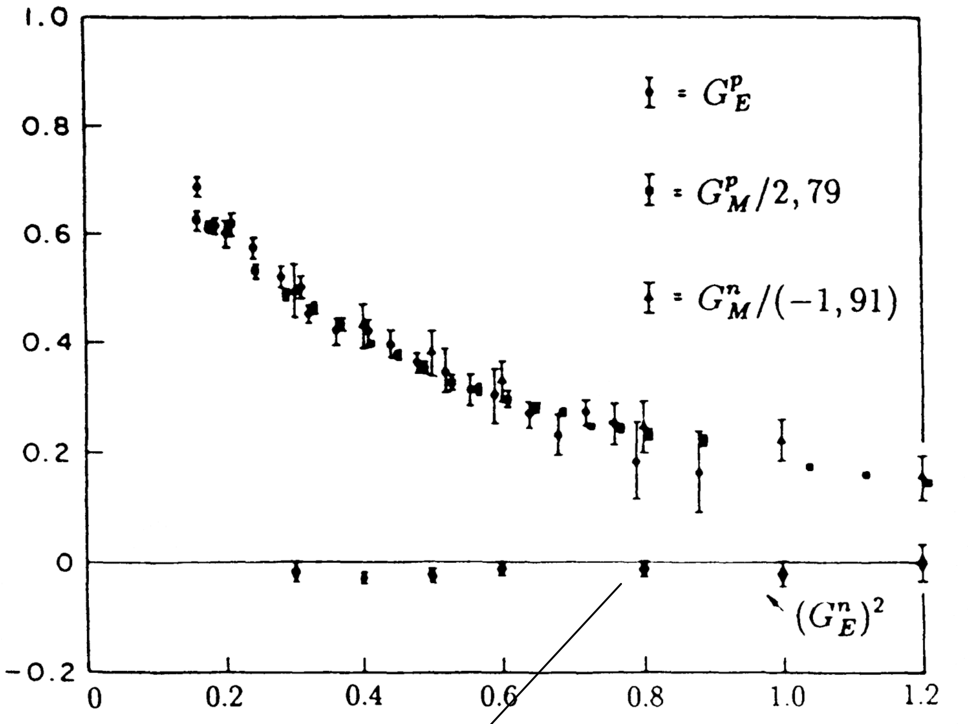
$G_M(Q^2)$ magnetic Formfactor

$G_E^p(Q^2=0)=1 \qquad G_E^n(Q^2=0)=0$

$G_M^p(Q^2=0)=2.79 \qquad G_M^n(Q^2=0)=-1.91$

Measuring the cross-sections for different Q values one can extract the $G_{E/M}(Q^2)$ dependency

Electric and magnetic form factors of proton and neutron



Obtained from scattering experiments of e^- on d

$$G_E^p(Q^2) = \frac{G_M^n(Q^2)}{2.79} = \frac{G_M^n(Q^2)}{-1.91} = G^{Dipol}(Q^2)$$

$$G^{Dipol}(Q^2) = \left(1 + \frac{Q^2}{0.71(GeV/c)^2}\right)^{-2}$$

F(q^2) dipol --> $\rho(r)=\rho(0)e^{-ar}$

a=4.27 fm^{-1}

For the proton radius:

$$\langle r^2 \rangle_{Dipol} = -6\hbar^2 \frac{dG^{Dipol}(Q^2)}{dQ^2}\bigg|_{Q^2=0} = \frac{12}{a^2} = 0.66\, fm^2$$

$$\sqrt{\langle r^2 \rangle_{Dipol}} = 0.81\, fm$$

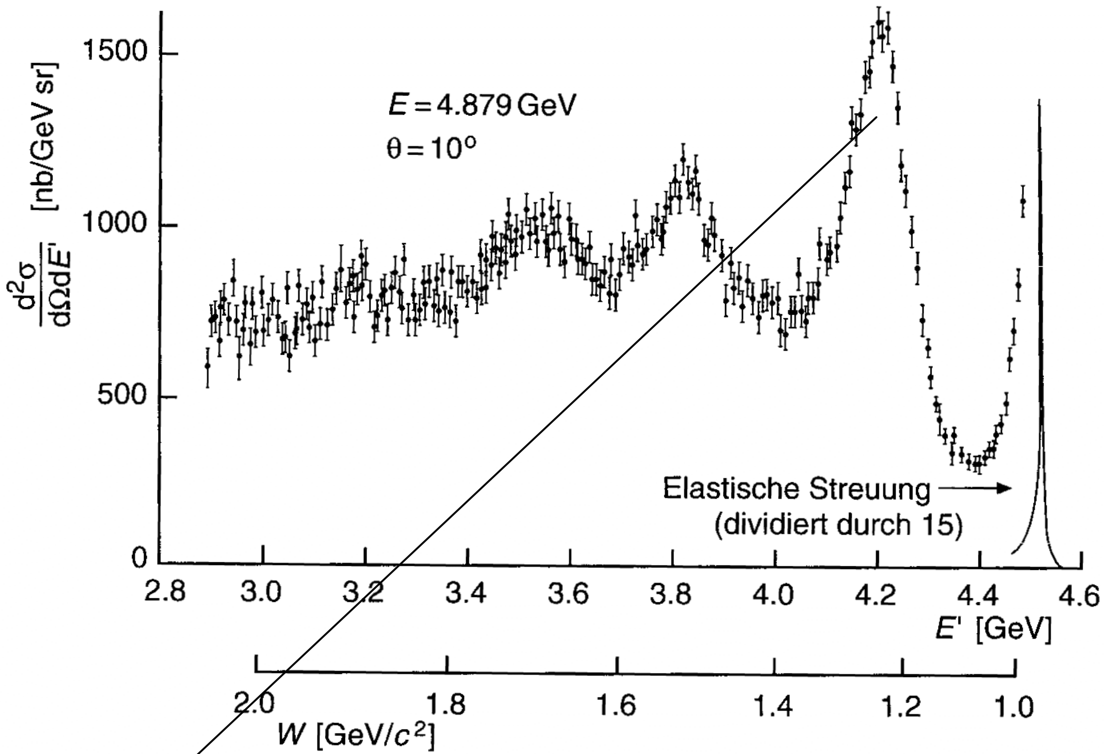
The mean squared charge radius for the neutron can be determined using the scattering of slow neutrons coming from a reactor to atomic electrons. In this way one obtains:

$$-6\hbar^2 \frac{dG_E^n(Q^2)}{dQ^2}\bigg|_{Q^2=0} = -0.117 \pm 0.002\, fm^2$$

$$\sqrt{\langle r^2 \rangle_n} = 0.10 \pm 0.01\, fm$$

Which means that inside the neutron we have charge constituents (quarks) which also have a magnetic momentum

Inelastic electron-proton scattering



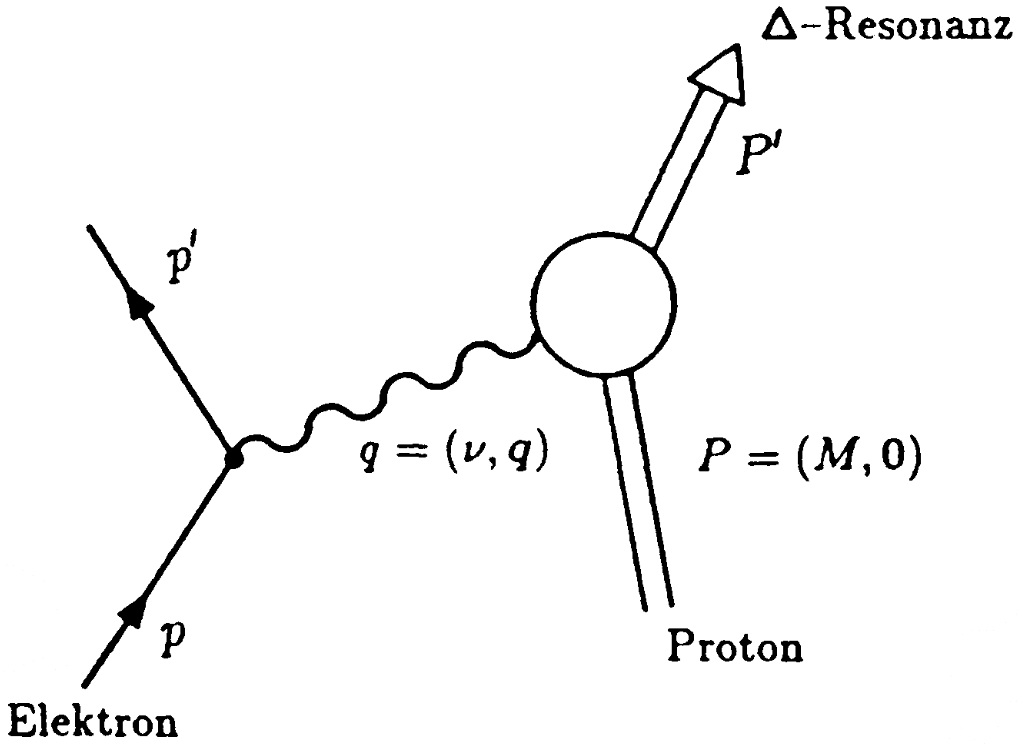
Povh et al., „Particles & nuclei"

$W^2c^2=P'^2=(P+q)^2$ P,P'= incoming and outcoming e^- momentu, q= momentum of the exchanged photons

Δ+: M=1232 MeV/c^2, Γ=100MeV

Exciting the delta resonance by inelastic ep scattering

W<2.5 GeV



Deep inelastic scattering (Hadron Production)

W>2.5 GeV