# Outline: Nuclear triaxial shape


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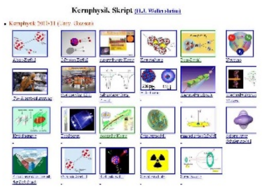
web-page: https://web-docs.gsi.de/~wolle/ and click on

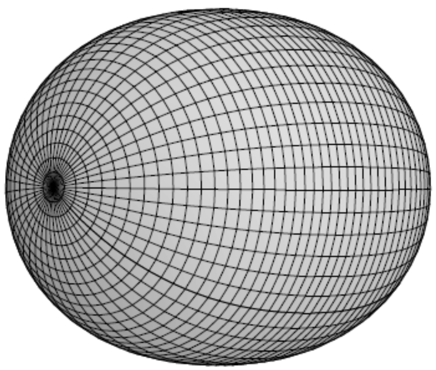



1. (β,γ) - coordinates
2. moment of inertia
3. triaxial rotor model
4. reduced transition probabilities (W, Os, Pt, Hg)

# Shape parameterization

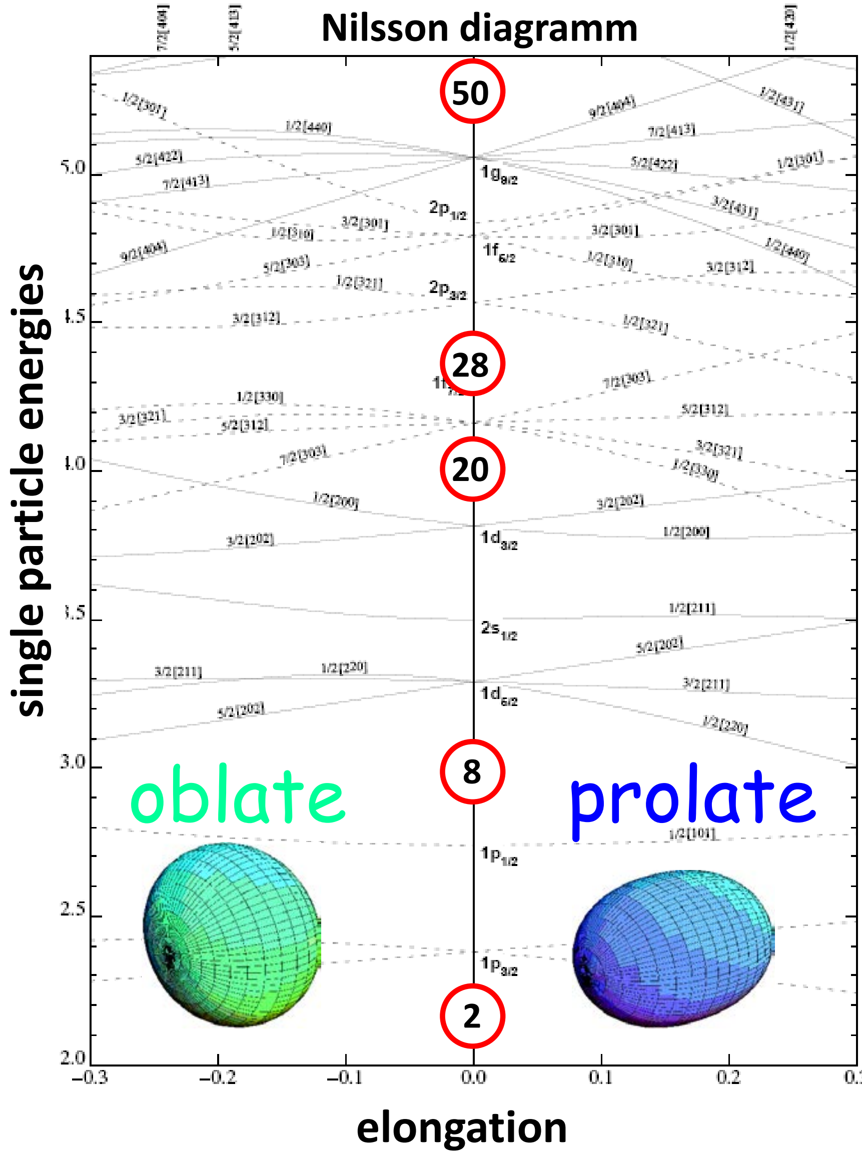
$$R(\theta,\phi) = R_0 \cdot \left[ 1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu} \cdot Y_{\lambda\mu}(\theta,\phi) \right]$$

axially symmetric **quadrupole**



**λ=2**

$\alpha_{20} \neq 0,\ \alpha_{2\pm1} = \alpha_{2\pm2} = 0$



# Quadrupole deformation (λ=2)

$$R(\theta,\phi) = R_0 \cdot \left[1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu} \cdot Y_{\lambda\mu}(\theta,\phi)\right]$$

There are five independent real parameters,

- $\alpha_{20}$ indicates the stretching of the 3-axis with respect to the 1- and 2-axes
- $\alpha_{22}$ determines the difference in length between the 1- and 2-axes
- three Euler angles, which determine the orientation of the principle axis system (1,2,3) with respect to the laboratory frame (x,y,z)

Hill - Wheeler introduced the (β, γ) – parameters:

$$a_{20} = \beta_2 \cos\gamma$$

$$a_{22} = \frac{1}{\sqrt{2}} \beta_2 \sin\gamma$$

# Quadrupole deformation (λ=2)

$$R(\theta,\phi) = R_0 \cdot \left\{1 + \beta \cdot \cos\gamma \cdot Y_{20}(\theta,\phi) + \frac{1}{\sqrt{2}} \cdot \beta \cdot \sin\gamma \cdot [Y_{22}(\theta,\phi) + Y_{2-2}(\theta,\phi)]\right\}$$

$$R(\theta,\phi) = R_0 \cdot \left\{1 + \beta \cdot \sqrt{\frac{5}{16\pi}}\left[\cos\gamma \cdot \left(3 \cdot \cos^2\theta - 1\right) + \sqrt{3} \cdot \sin\gamma \cdot \sin^2\theta \cdot \cos 2\phi\right]\right\}$$

Consider the nuclear shapes in the principal axis system (1, 2, 3) ≡ (x´, y´, z´)

$$R_1 = R_{x'} = R\left(\frac{\pi}{2},0\right) = R_0 \cdot \left\{1 + \beta \cdot \sqrt{\frac{5}{16\pi}}\left[-\cos\gamma + \sqrt{3} \cdot \sin\gamma\right]\right\}$$

$$R_2 = R_{y'} = R\left(\frac{\pi}{2},\frac{\pi}{2}\right) = R_0 \cdot \left\{1 + \beta \cdot \sqrt{\frac{5}{16\pi}}\left[-\cos\gamma - \sqrt{3} \cdot \sin\gamma\right]\right\}$$

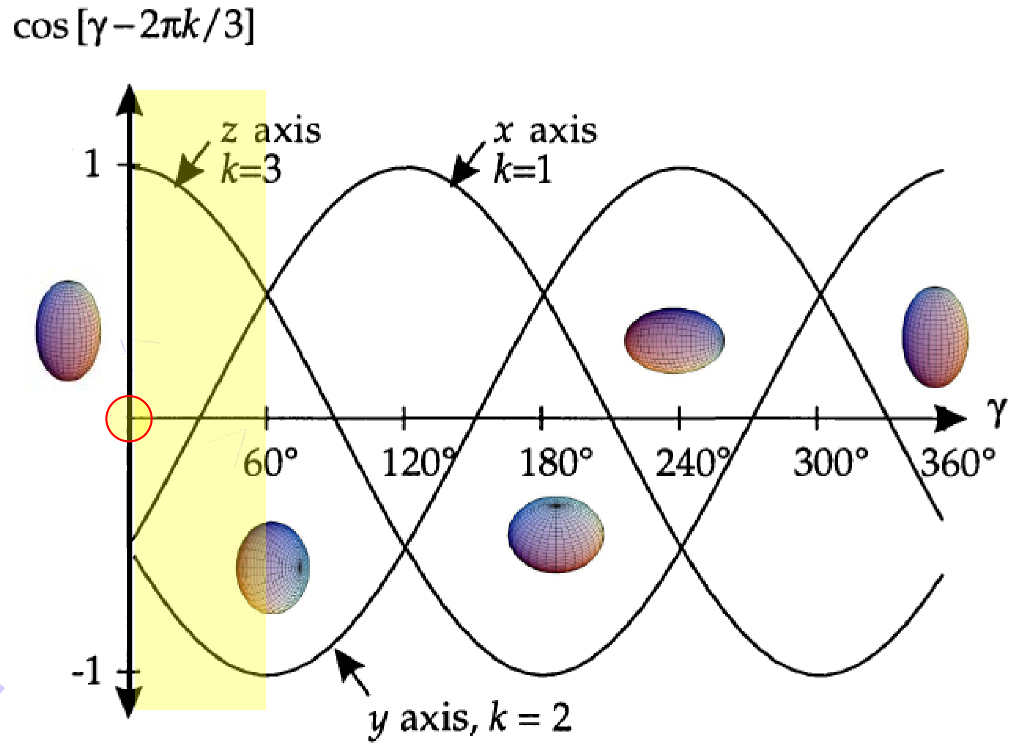
$$R_3 = R_{z'} = R(0,0) = R_0 \cdot \left\{1 + \beta \cdot \sqrt{\frac{5}{16\pi}} \cdot 2 \cdot \cos\gamma\right\}$$

$$R_k(\theta,\phi) = R_0 \cdot \left\{1 + \beta \cdot \sqrt{\frac{5}{4\pi}} \cdot \cos\left(\gamma - \frac{2\pi \cdot k}{3}\right)\right\} \quad for \;\; k = 1,2,3$$

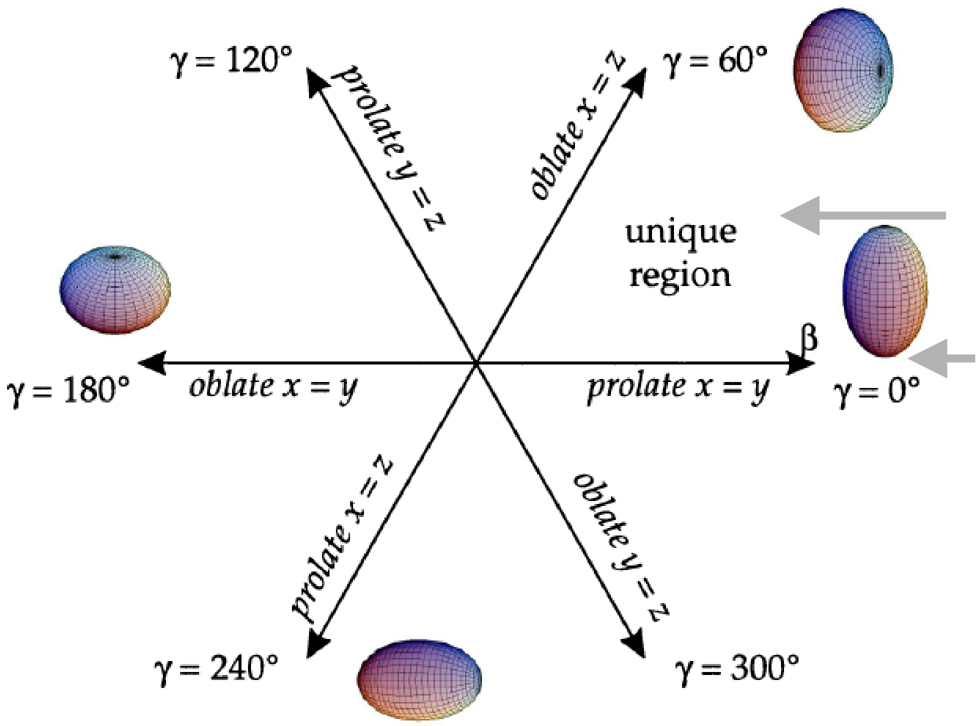
# (β,γ) coordinates

$$R_k(\theta,\phi) = R_0 \cdot \left\{1 + \beta \cdot \sqrt{\frac{5}{4\pi}} \cdot \cos\left(\gamma - \frac{2\pi \cdot k}{3}\right)\right\} \quad for \;\; k = 1,2,3$$

- At $\gamma = 0^0$ the nucleus is elongated along the z´axis, but the x´and y´axes are equal (**prolate shape** for x´= y´)
- As we increase $\gamma$, the x´axis grows at the expense of the y´and z´axes through a region of **triaxial shapes** with three unequal axis, until axial symmetry is again reached at $\gamma = 60^0$, but now with the z´and x´axis equal in length. These two axes are longer than the y´axis (**oblate shape** for x´= z´)
- This pattern is **repeated: every 60$^0$** axial symmetry repeated and prolate and oblate shapes alternate.



# (β,γ) coordinates




The (β,γ) plane is divided into six equivalent parts by the symmetries:

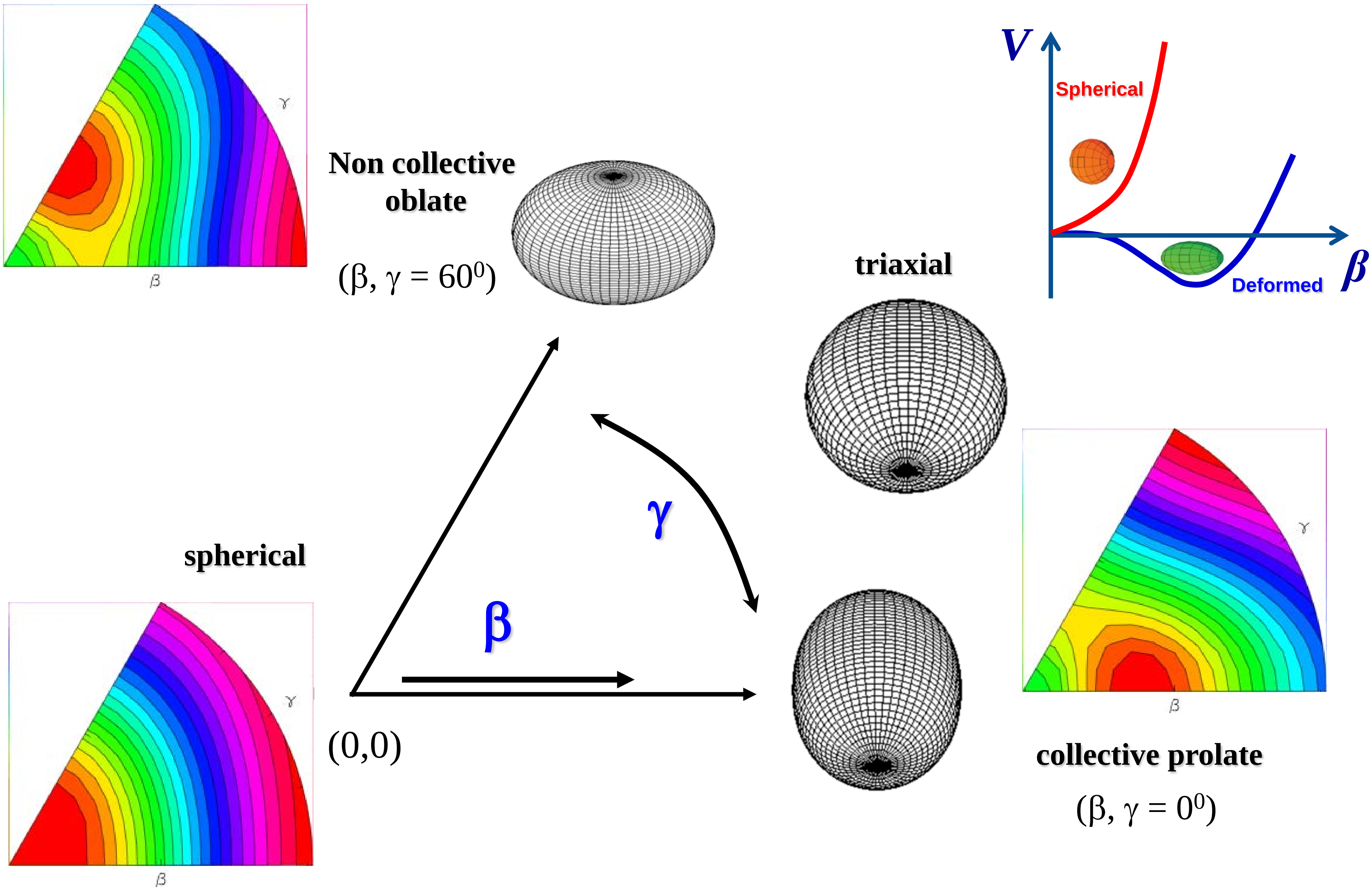
The sector $0^0$ and $60^0$ contains all shapes uniquely, i.e. **triaxial shapes**

The types of shapes encountered along the axis: e.g. **prolate x´= y´** implies prolate shapes with the z´axis as the long axis and the two other axis x´and y´equal.

⇨ various nuclear shapes – **prolate or oblate** – in the (β,γ) plane **are repeated every $60^0$**. Because the axis orientations are different, the associated Euler angles also differ.

**In conclusion, the same physical shape (including ist orientation in space) can be represented by different sets of deformation parameters (β,γ) and Euler angles!**

# (β,γ) coordinates



- **Rotational (deformed):**

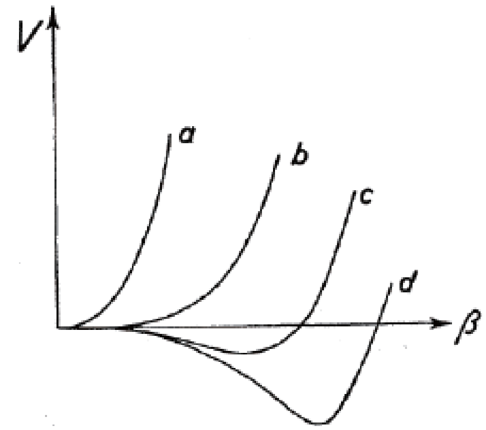
$$E_x(I) = \frac{I(I+1)\hbar^2}{2\cdot\mathfrak{I}}$$

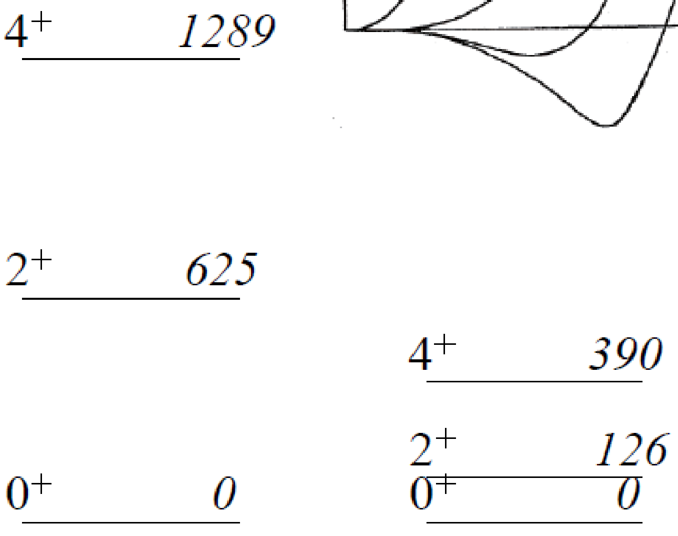
  – $E(4^+) / E(2^+) = 10/3$

- **Vibrational (spherical):**

$$E_n = n\cdot\hbar\omega_2$$

  – $E(4^+) / E(2^+) = 2$




**¹⁰⁸Te** **¹⁶⁰Er**

$$\frac{E(4^+)}{E(2^+)} = \quad \mathbf{2.1} \quad\quad \mathbf{3.1}$$

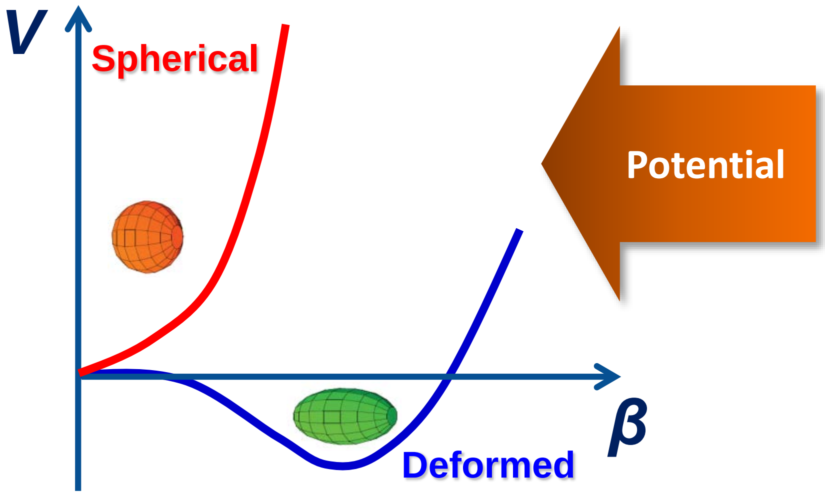
# Classical collective Hamiltonian of Bohr-Mottelson for quadrupole deformation

$$H_{coll} = T_{vib} + T_{rot} + V_{coll} = \frac{1}{2}\sum_{\lambda\mu} B_\lambda \left|\dot{\alpha}_{\lambda\mu}\right|^2 + \frac{1}{2}\sum_{k=1}^{3} \Im_k \omega_k^2 + \frac{1}{2}\sum_{\lambda\mu} C_\lambda \left|\alpha_{\lambda\mu}\right|^2$$

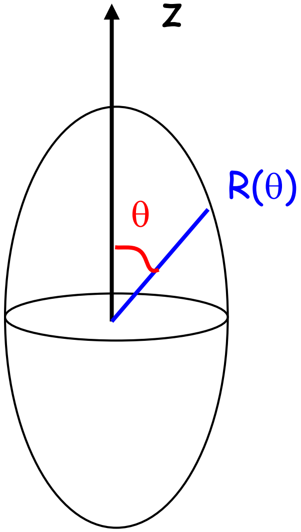
Quadrupole ($\lambda$=2) motion

$$H = \frac{1}{2} B \cdot \left(\dot{\beta}^2 + \beta^2 \cdot \dot{\gamma}^2\right) + \frac{1}{2}\sum_k \Im_k \cdot \omega_k^2 + \frac{1}{2} C \cdot \beta^2$$

where ($\beta$, $\gamma$) parameters have been used.

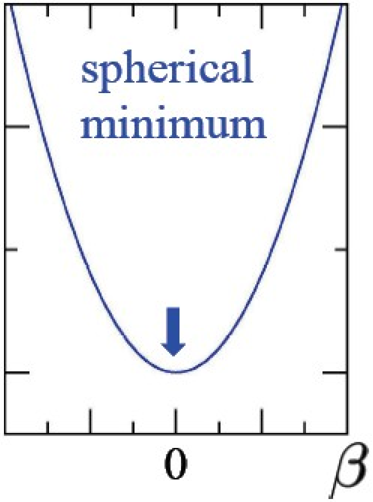


# Moment of inertia




$$R(\theta) = R_0 \cdot \left[1 + \beta \cdot Y_{20}(\theta)\right]$$

$$\beta = \frac{4}{3} \cdot \sqrt{\frac{\pi}{5}} \cdot \frac{R(0^0) - R(90^0)}{R_0} \cong 1.05 \cdot \frac{\Delta R}{R_0}$$



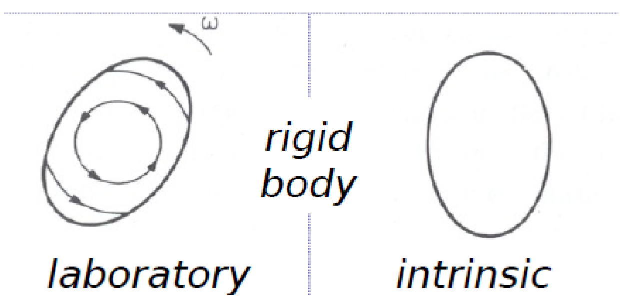

Rigid body moment of inertia:

$$\Im_R = \iiint r^2 \cdot \rho(r) \cdot r^2 dr \sin\theta \, d\theta \, d\phi$$

$$\Im_R = \frac{2}{5} M \; R_o^2 \; (1 + 0.32\beta)$$

Irrotational flow moment of inertia:

$$\Im_F = \frac{9}{8\pi} \; M \; R_o^2 \; \beta^2$$

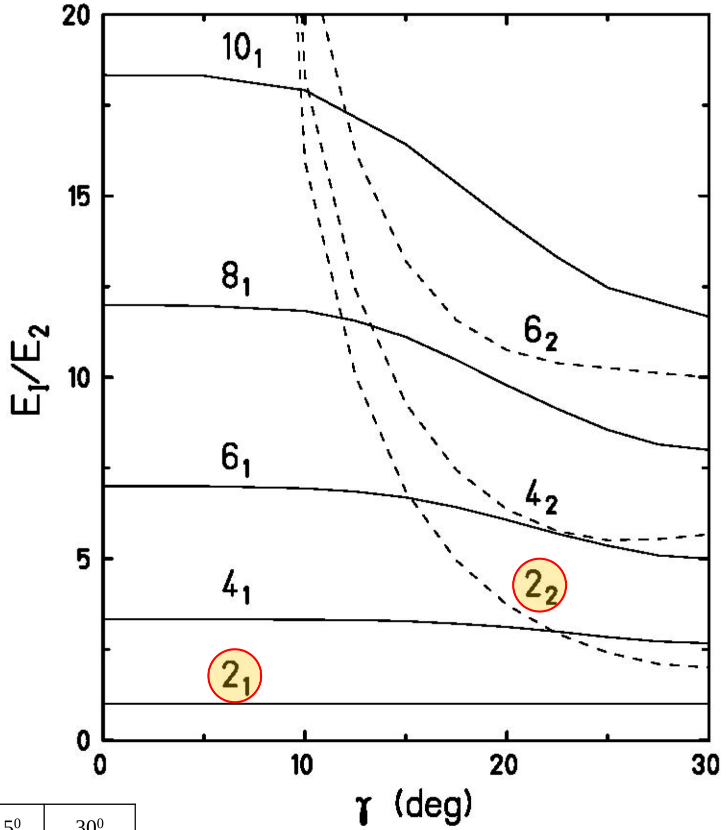


# $2_2^+$ energy and estimate of γ-deformation parameter

**rigid triaxial rotor model**

$$\frac{E(2_2)}{E(2_1)} = \frac{3+\sqrt{9-8\cdot\sin^2(3\gamma)}}{3-\sqrt{9-8\cdot\sin^2(3\gamma)}} \geq 2$$

*Davydov and Filippov, Nucl. Phys. 8, 237 (1958)*




| γ | $0^0$ | $5^0$ | $10^0$ | $12.5^0$ | $15^0$ | $17.5^0$ | $20^0$ | $22.5^0$ | $25^0$ | $27.5^0$ | $30^0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $E(4_1)/E(2_1)$ | 3.33 | 3.33 | 3.32 | 3.31 | 3.28 | 3.21 | 3.12 | 2.99 | 2.84 | 2.72 | 2.67 |
| $E(6_1)/E(2_1)$ | 7.00 | 7.00 | 6.94 | 6.85 | 6.69 | 6.42 | 6.07 | 5.69 | 5.36 | 5.09 | 5.00 |
| $E(8_1)/E(2_1)$ | 12.00 | 11.97 | 11.83 | 11.56 | 11.11 | 10.48 | 9.78 | 9.13 | 8.55 | 8.15 | 8.00 |
| $E(10_1)/E(2_1)$ | 18.33 | 18.31 | 17.91 | | 16.42 | | 14.30 | 13.31 | 12.47 | | 11.67 |
| $E(2_2)/E(2_1)$ | ∞ | 65.16 | 15.94 | 10.04 | 6.85 | 4.95 | 3.73 | 2.93 | 2.41 | 2.10 | 2.00 |
| $E(4_2)/E(2_1)$ | ∞ | 67.50 | 18.28 | 12.41 | 9.27 | 7.44 | 6.36 | 5.76 | 5.51 | 5.54 | 5.67 |
| $E(6_2)/E(2_1)$ | ∞ | 71.17 | 22.00 | 16.20 | 13.19 | 11.57 | 10.75 | 10.39 | 10.26 | 10.12 | 10.00 |

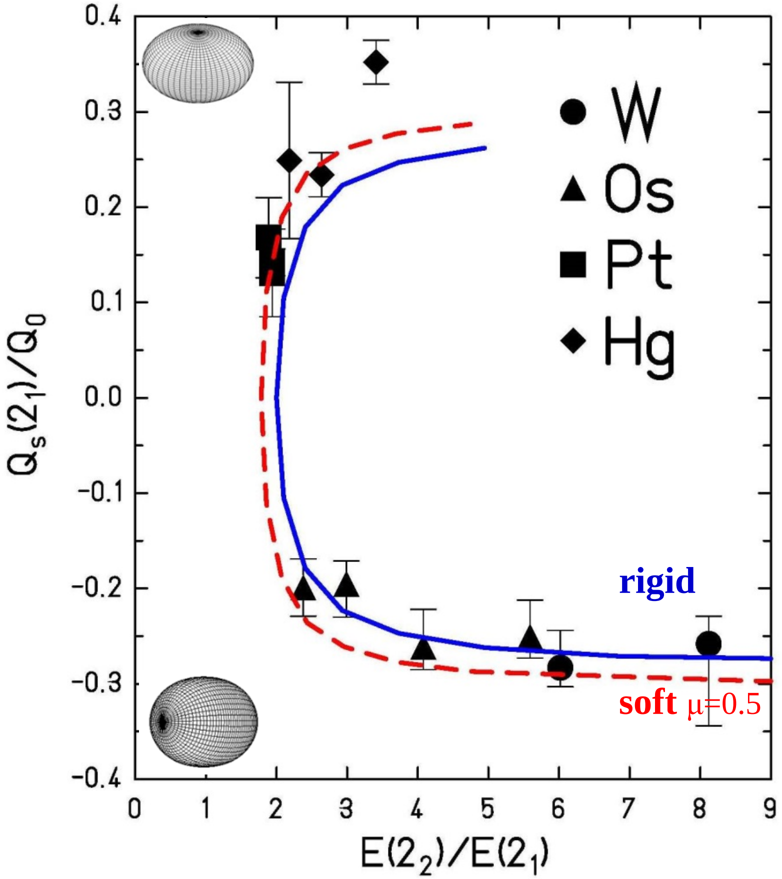
# Prolate – oblate shape transition

**rigid triaxial rotor model**

$$\frac{Q_s(2_1)}{Q_0} = -\frac{6 \cdot \cos(3\gamma)}{7 \cdot \sqrt{9 - 8 \cdot \sin^2(3\gamma)}}$$

*Davydov and Filippov, Nucl. Phys. 8, 237 (1958)*

$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I \cdot (2I-1)}{(I+1) \cdot (2I+1) \cdot (2I+3)}} \cdot \frac{\langle I \| M(E2) \| I \rangle}{\langle 2_1 \| M(E2) \| 0_1 \rangle}$$




| γ | $0^0$ | $10^0$ | $15^0$ | $20^0$ | $22.5^0$ | $25^0$ | $27.5^0$ | $30^0$ |
|---|---|---|---|---|---|---|---|---|
| $Q_s(2_1)/Q_0$ | -0.28 | -0.28 | -0.27 | -0.25 | -0.22 | -0.18 | -0.10 | 0.0 |

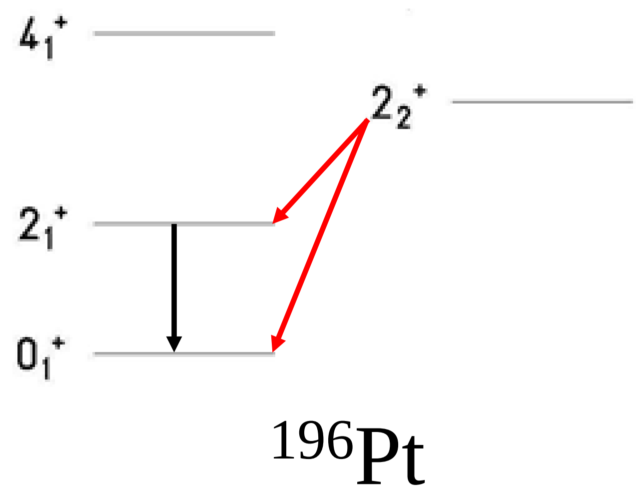
soft asymmetric rotor model: $\gamma \rightarrow \gamma_{eff} = \sqrt{\Gamma^2 + \gamma_o^2}$ with $\Gamma = \left\{\langle 0 | (\gamma - \gamma_0)^2 | 0 \rangle\right\}^{1/2}$ $\mu = \left\{\frac{\langle 0 | (\beta - \beta_0)^2 | 0 \rangle}{\beta_0^2}\right\}^{1/2}$

# $2_2^+$ B(E2)-values and estimate of γ-deformation parameter

**rigid triaxial rotor model**

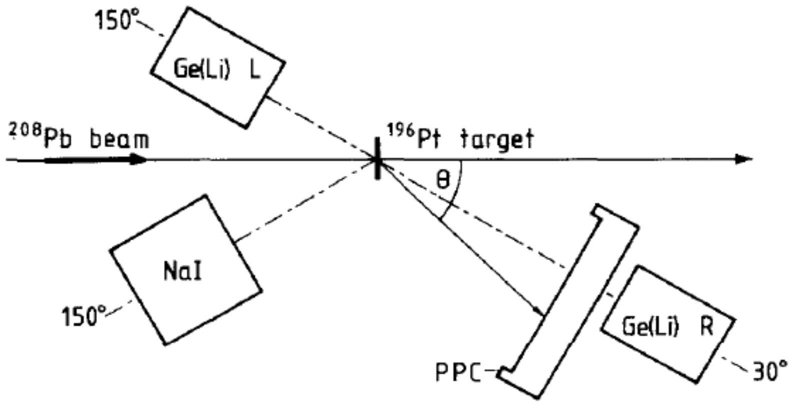
$$\frac{B(E2;2_2\rightarrow 0)}{B(E2;2_1\rightarrow 0)}=\frac{1-\dfrac{3-2\sin^2(3\gamma)}{\sqrt{9-8\sin^2(3\gamma)}}}{1+\dfrac{3-2\sin^2(3\gamma)}{\sqrt{9-8\sin^2(3\gamma)}}}$$

$$\frac{B(E2;2_2\rightarrow 2_1)}{B(E2;2_1\rightarrow 0)}=\frac{\dfrac{20}{7}\,\dfrac{\sin^2(3\gamma)}{9-8\sin^2(3\gamma)}}{1+\dfrac{3-2\sin^2(3\gamma)}{\sqrt{9-8\sin^2(3\gamma)}}}$$

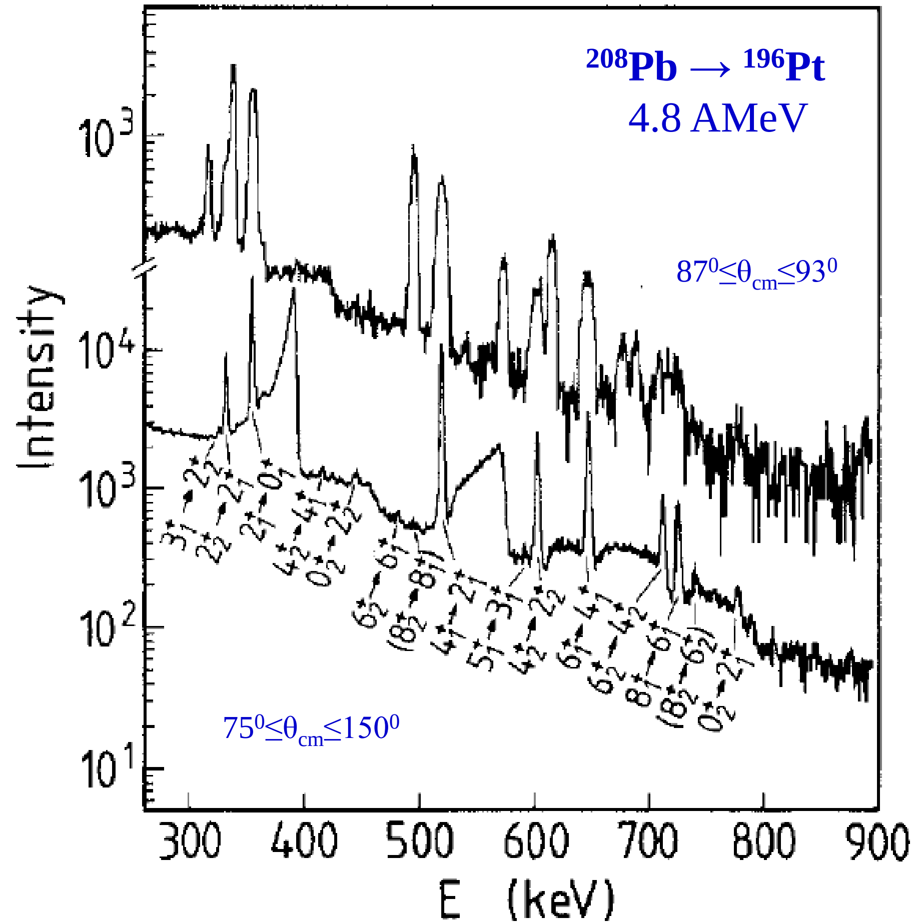



| γ | $0^0$ | $5^0$ | $10^0$ | $15^0$ | $20^0$ | $25^0$ | $30^0$ |
|---|---|---|---|---|---|---|---|
| $B(E2;2_2\rightarrow0)/B(E2;2_1\rightarrow0)$ | 0 | .0075 | .0288 | .0560 | .0718 | .0445 | 0 |
| $B(E2;2_2\rightarrow0)/B(E2;2_1\rightarrow0)$ | 0 | .0111 | .0525 | .1510 | .3826 | .9058 | 1.43 |
| $B(E2;2_2\rightarrow2)/B(E2;2_2\rightarrow0)$ | 1.43 | 1.49 | 1.70 | 2.70 | 5.35 | 20.6 | ∞ |

# Triaxiality and γ-softness in $^{196}$Pt

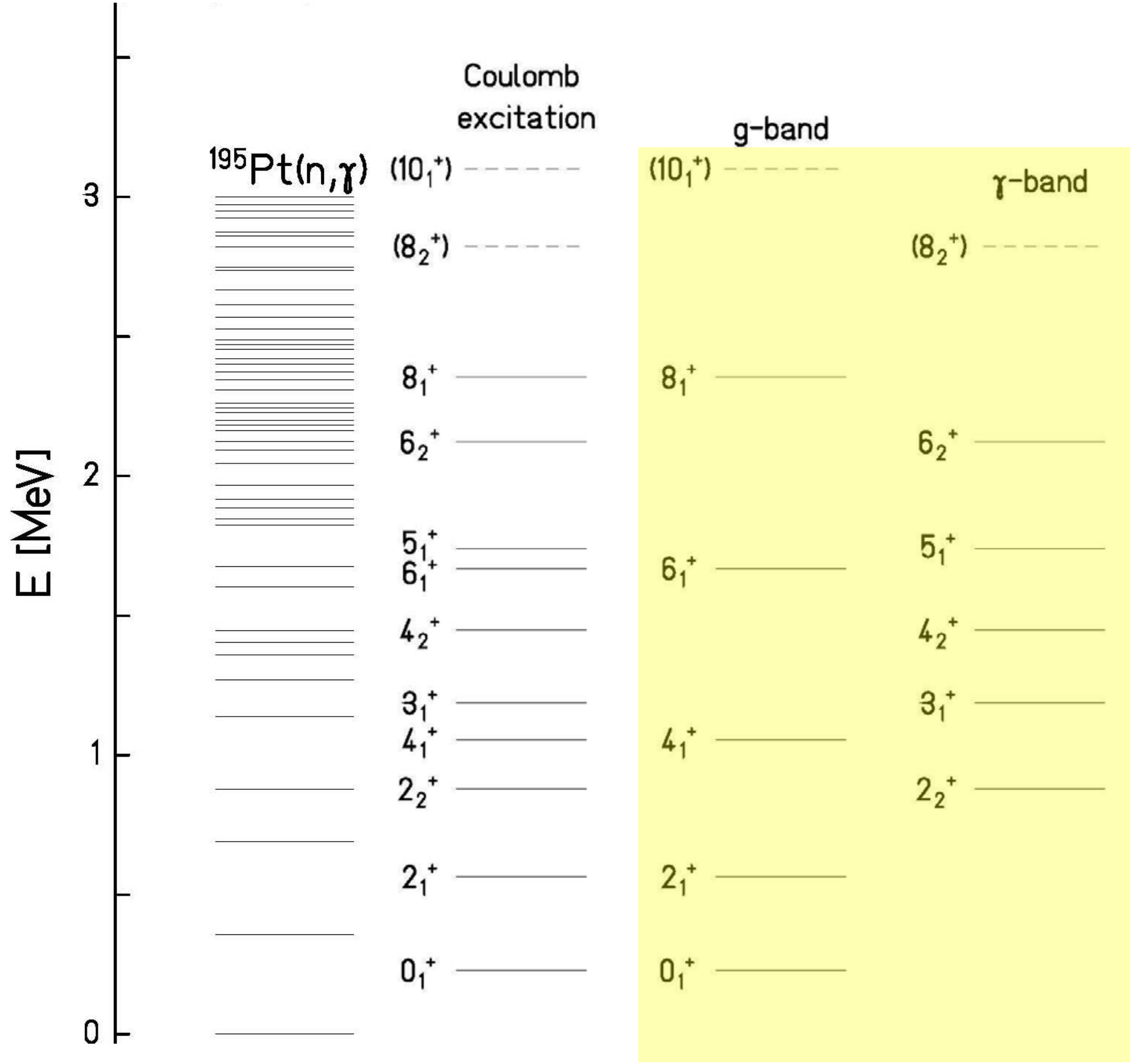



position-sensitive
parallel plate counter
(Doppler correction)




*A. Mauthofer et al., Z.Phys. A336, 263 (1990)*

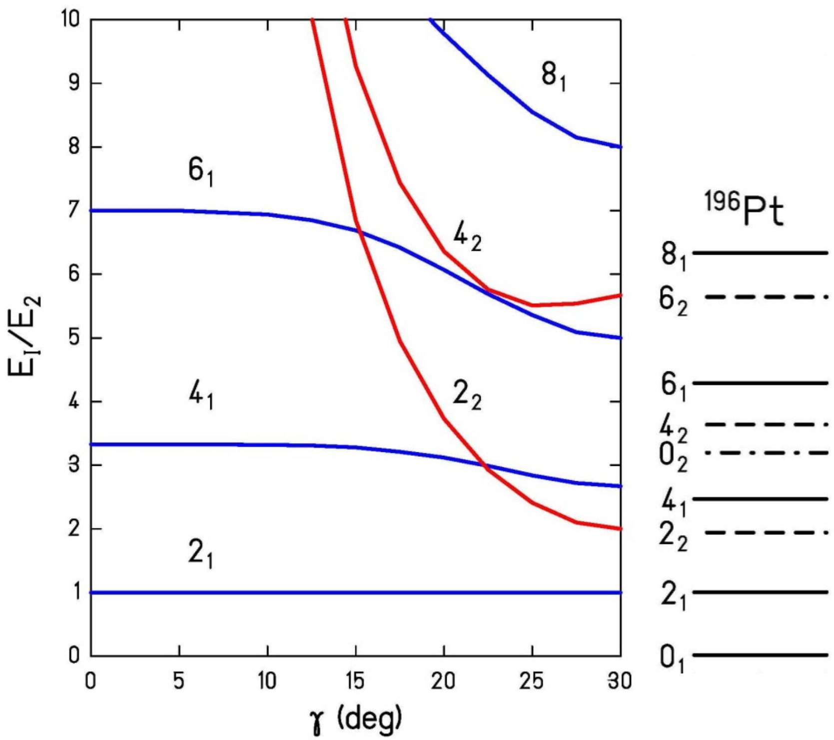
# Level scheme of $^{196}$Pt



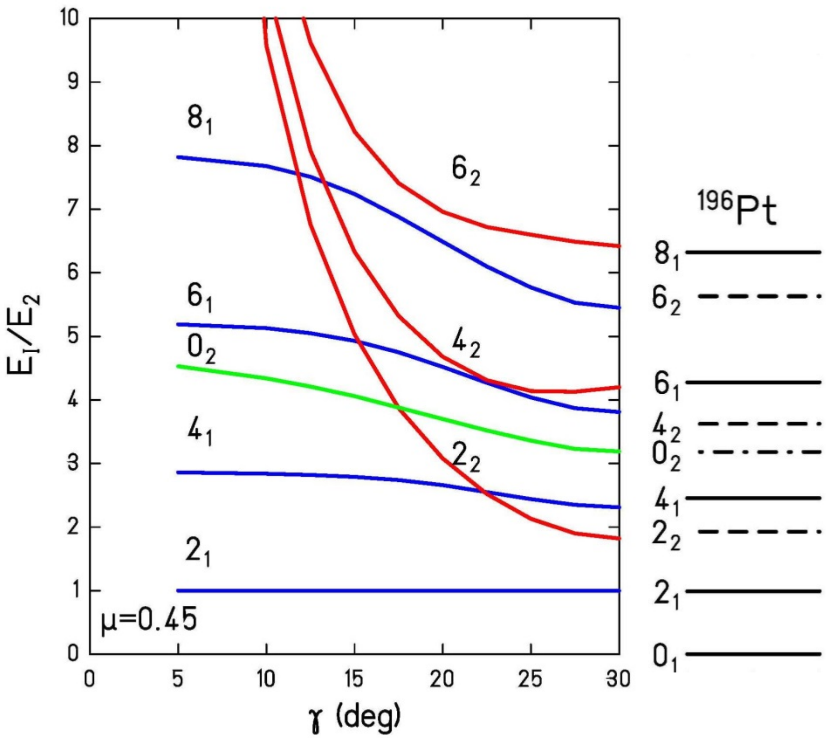

*A. Mauthofer et al., Z.Phys. A336, 263 (1990)*

# Level scheme of $^{196}$Pt

Comparison with
**rigid** asymmetric rotor model



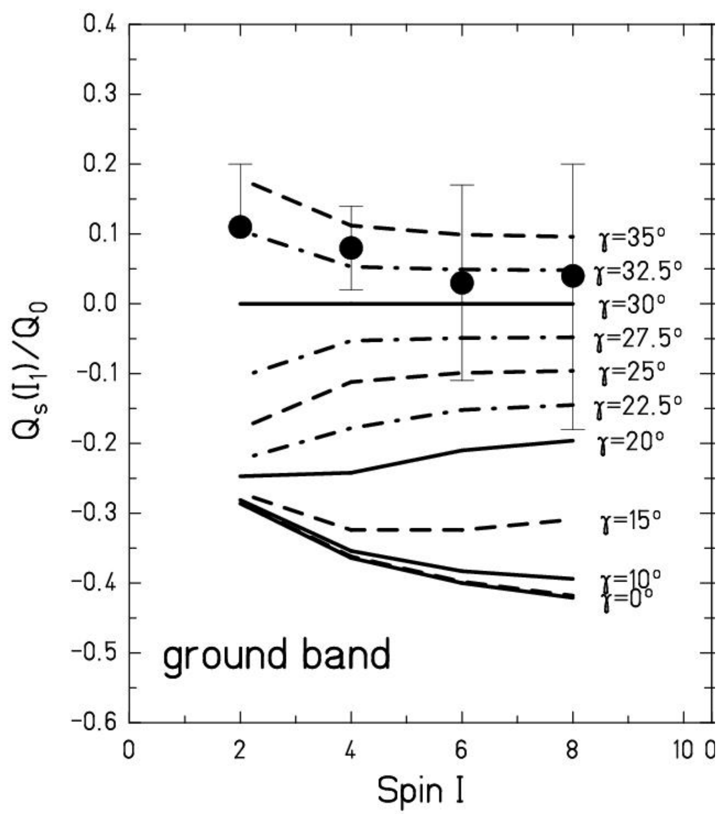

Comparison with
**soft** asymmetric rotor model

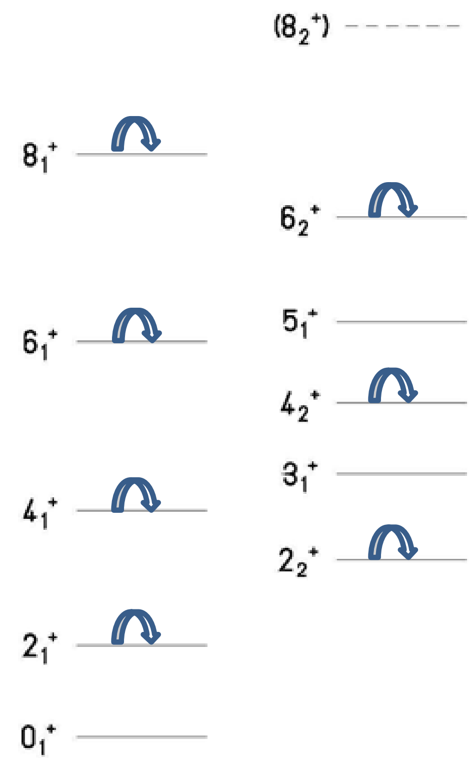


# Spin dependence of the spectroscopic quadrupole moment

$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I\cdot(2I-1)}{(I+1)\cdot(2I+1)\cdot(2I+3)}}\frac{\langle I\|M(E2)\|I\rangle}{\langle 2_1\|M(E2)\|0_1\rangle}$$

$Q_0 = 3.87(7)\ [b]$ $\qquad Q_0 = \frac{3\cdot Z\cdot R_0^2}{\sqrt{5\pi}}\cdot\beta$ $\qquad \beta = 0.135(2)$

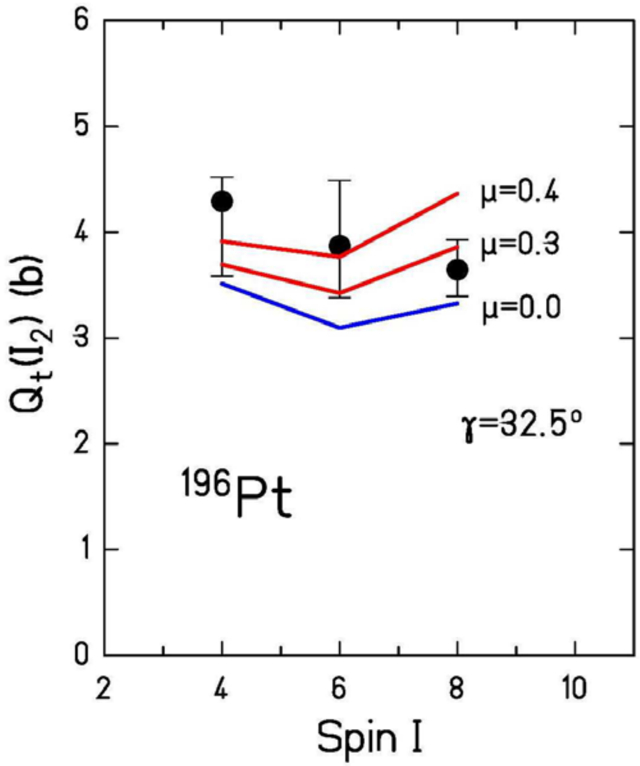



*A. Mauthofer et al., Z.Phys. A336, 263 (1990)*

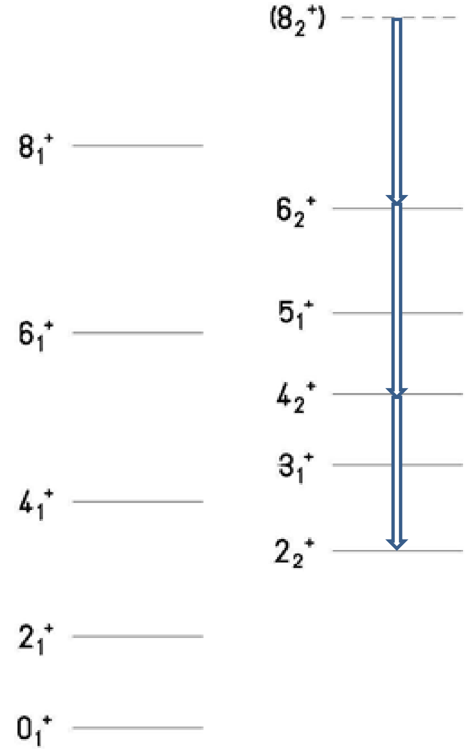
# Transition quadrupole moment in the γ-band

$$Q_t(I_2) = \sqrt{\frac{(2I-2)\cdot(2I-1)\cdot I}{3\cdot(I+1)\cdot(I+2)\cdot(I-2)\cdot(I-3)}}\cdot\sqrt{\frac{16\pi}{5}}\cdot\langle I_2-2\|M(E2)\|I_2\rangle$$



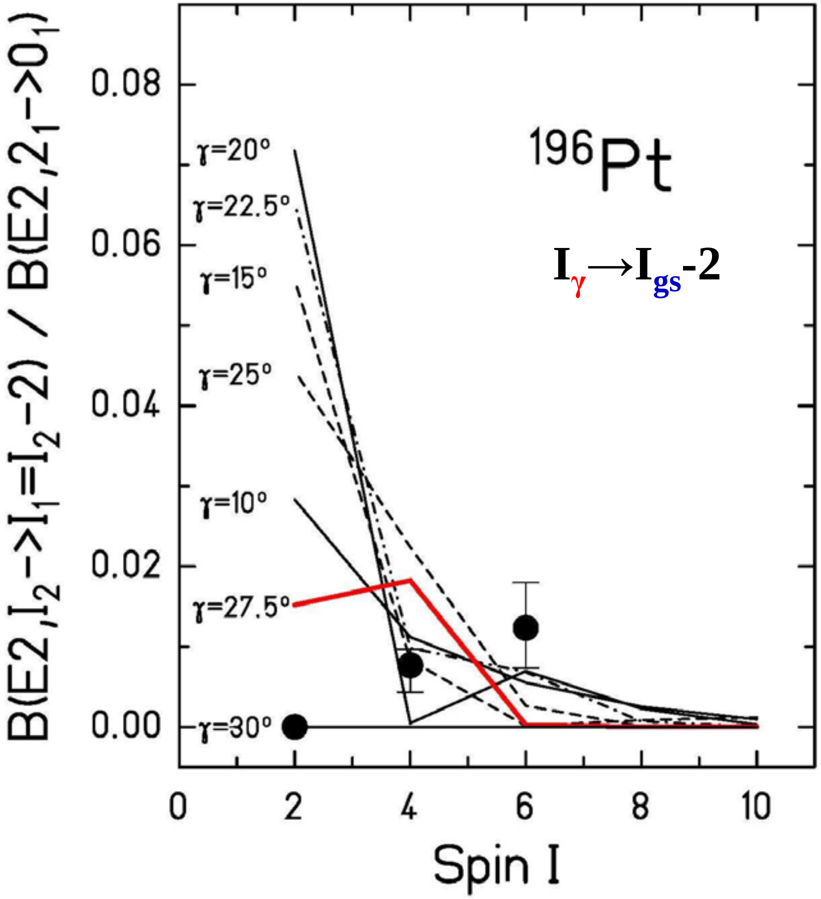

soft ARM

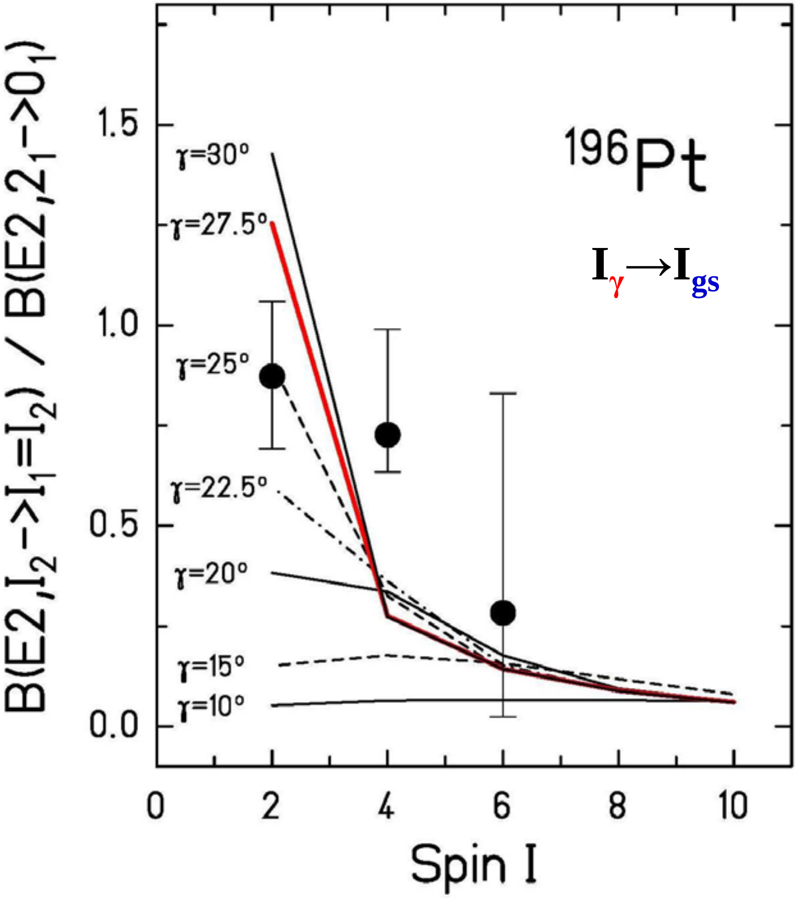
rigid ARM




*A. Mauthofer et al., Z.Phys. A336, 263 (1990)*

# B(E2)-values connecting the γ- and gs-band




$$\frac{B(E2;2_2 \to 0)}{B(E2;2_1 \to 0)} = \frac{1 - \frac{3 - 2\sin^2(3\gamma)}{\sqrt{9 - 8\sin^2(3\gamma)}}}{1 + \frac{3 - 2\sin^2(3\gamma)}{\sqrt{9 - 8\sin^2(3\gamma)}}}$$

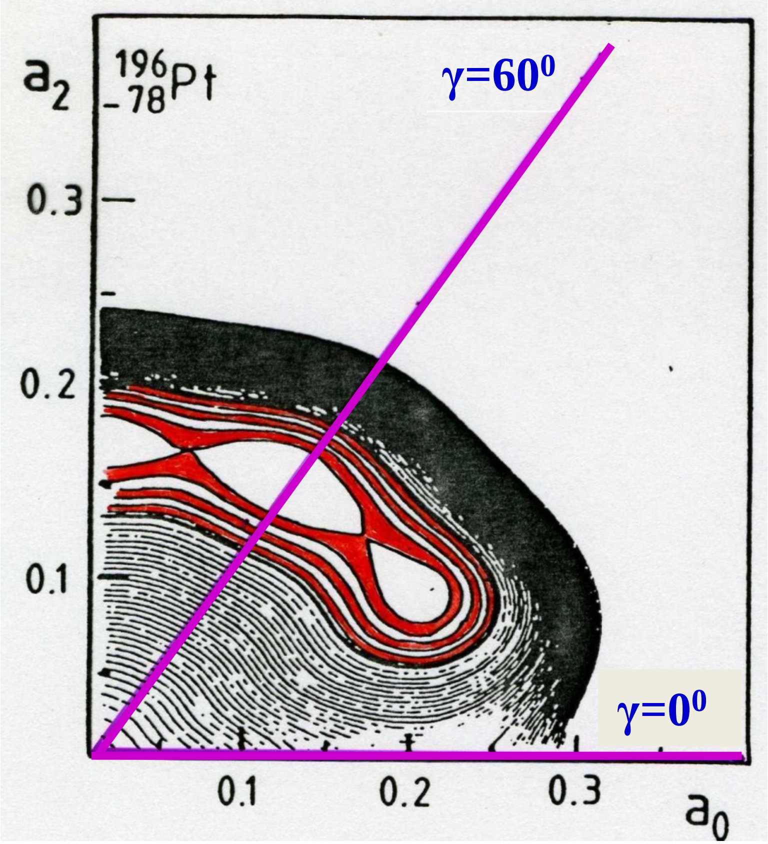
$$\frac{B(E2;2_2 \to 2_1)}{B(E2;2_1 \to 0)} = \frac{\frac{20}{7}\,\frac{\sin^2(3\gamma)}{9 - 8\sin^2(3\gamma)}}{1 + \frac{3 - 2\sin^2(3\gamma)}{\sqrt{9 - 8\sin^2(3\gamma)}}}$$

*A. Mauthofer et al., Z.Phys. A336, 263 (1990)*

# Interpretation of the collective properties in $^{196}$Pt

**14** energy levels and **22** E2 matrix elements can be described by the soft asymmetric rotor model assuming the following parameters:

$$\hbar^2/2\Im = 40.2\ keV \qquad \beta = 0.135 \qquad \gamma = 32.5^0 \qquad \mu = 0.35$$




$$a_0 = \beta \cdot \cos\gamma$$

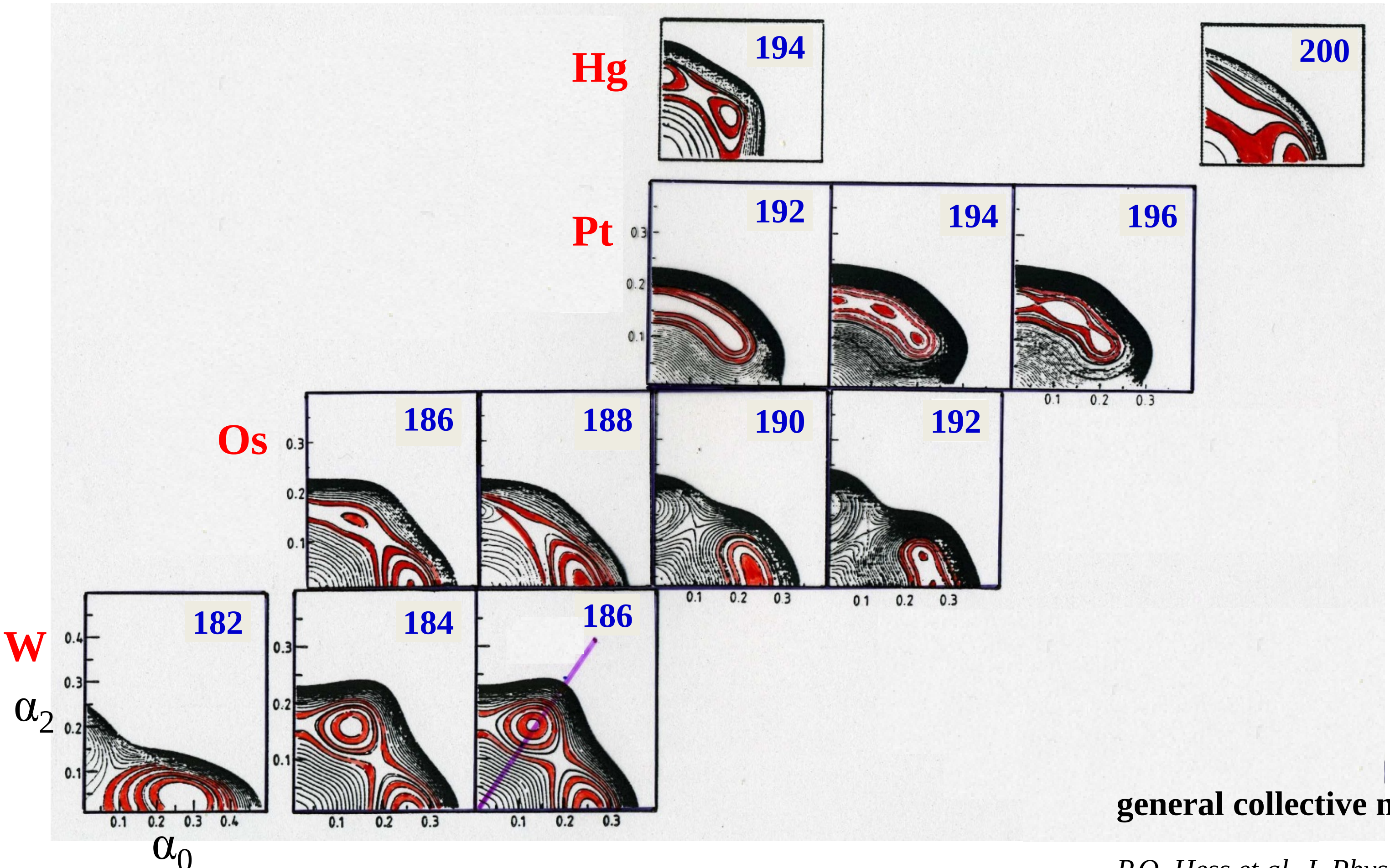
$$a_2 = \frac{1}{\sqrt{2}}\beta \cdot \sin\gamma$$

*P.O. Hess et al. J.Phys. G7 (1981), 737*

# Potential energy surfaces of the W-Os-Pt-Hg chain of isotopes

$$a_0 = \beta \cdot \cos\gamma$$

$$a_2 = \frac{1}{\sqrt{2}} \beta \cdot \sin\gamma$$

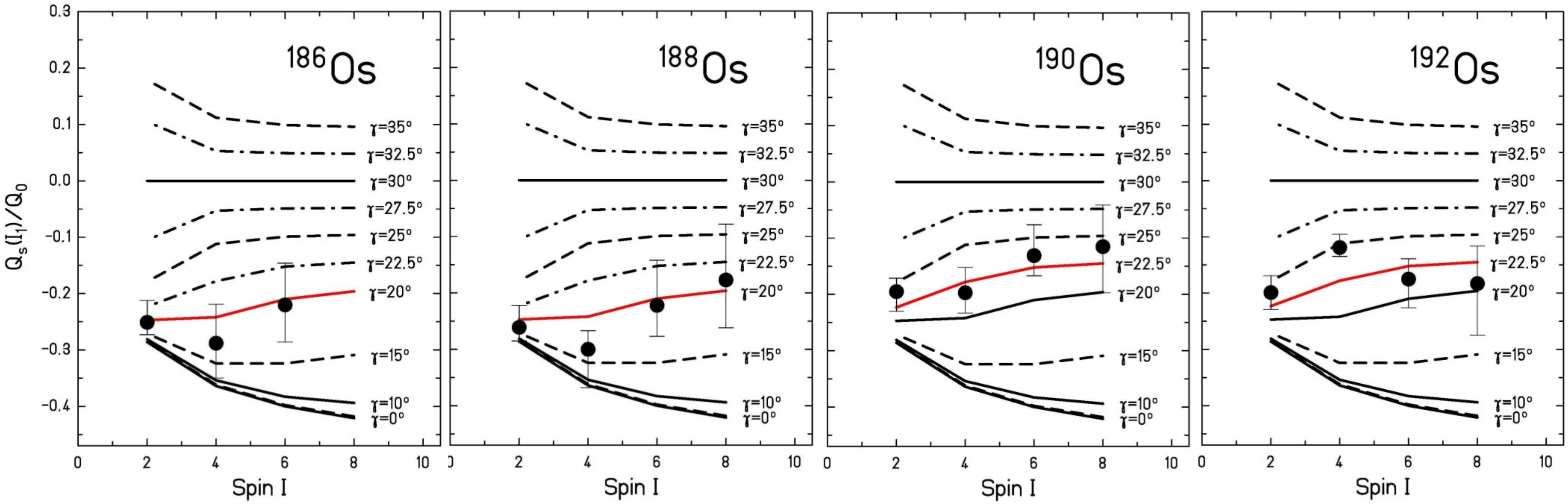



**general collective model**

*P.O. Hess et al. J. Phys. G7 (1981), 737*

# Spectroscopic quadrupole moments in the ground state band of Os-isotopes

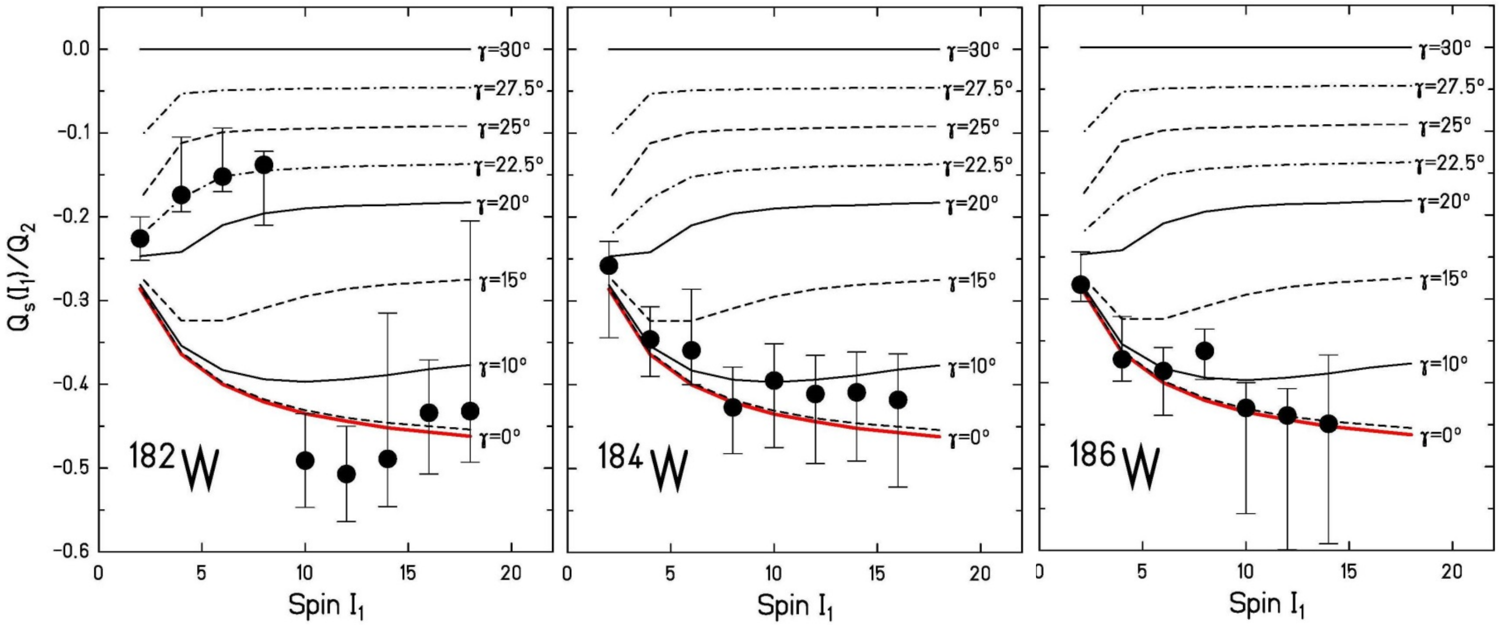
$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I\cdot(2I-1)}{(I+1)\cdot(2I+1)\cdot(2I+3)}}\cdot\frac{\langle I\|M(E2)\|I\rangle}{\langle 2_1\|M(E2)\|0_1\rangle}$$




*C.Y. Wu, D. Cline et al.; Ann. Rev. Nucl. Part Sci 36 (1986), 683*

# Spectroscopic quadrupole moments in the ground state band of W-isotopes

$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I\cdot(2I-1)}{(I+1)\cdot(2I+1)\cdot(2I+3)}}\cdot\frac{\langle I\|M(E2)\|I\rangle}{\langle 2_1\|M(E2)\|0_1\rangle}$$




$Q_0$ = 7.10 (38) b
β = 0.274 – 0.193

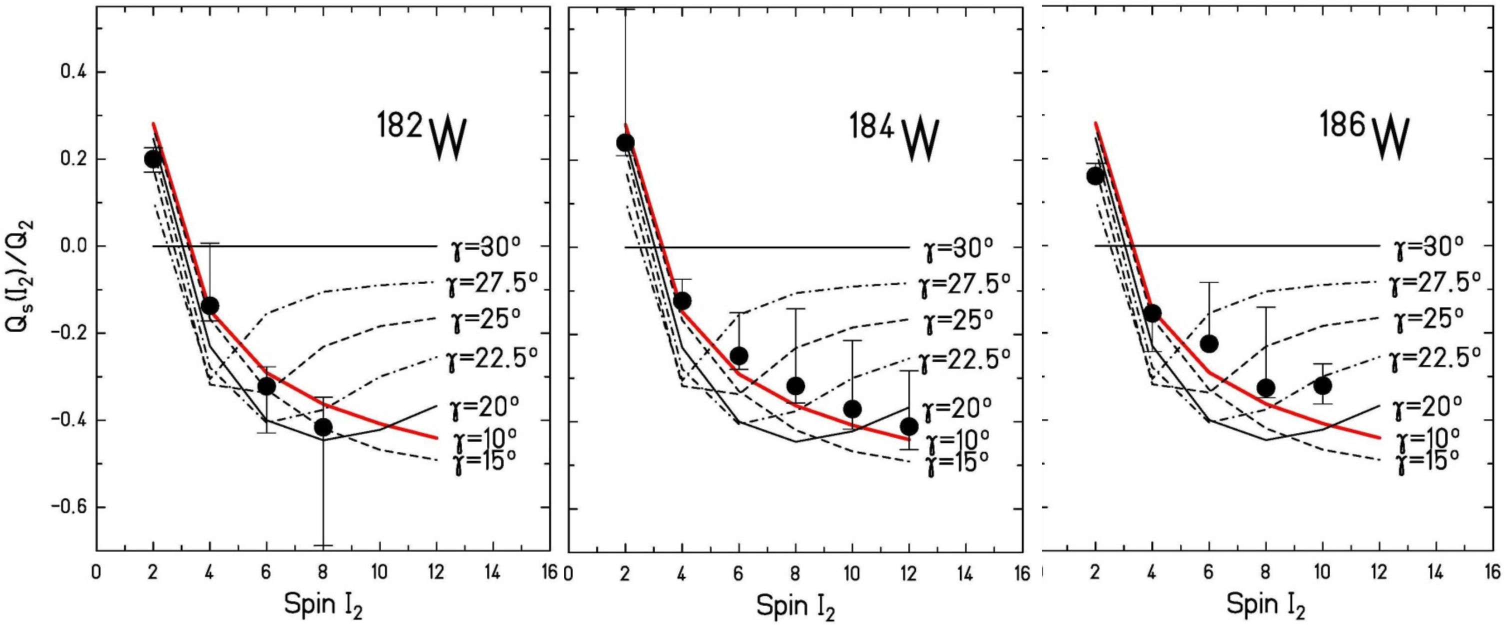
$Q_0$ = 6.72 (35) b
β = 0.258

$Q_0$ = 5.87 (29) b
β = 0.223

*R. Kulessa et al.; Phys. Lett B218 (1989), 421*

# Spectroscopic quadrupole moments in the gamma band of W-isotopes

$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I\cdot(2I-1)}{(I+1)\cdot(2I+1)\cdot(2I+3)}}\cdot\frac{\langle I\|M(E2)\|I\rangle}{\langle 2_1\|M(E2)\|0_1\rangle}$$



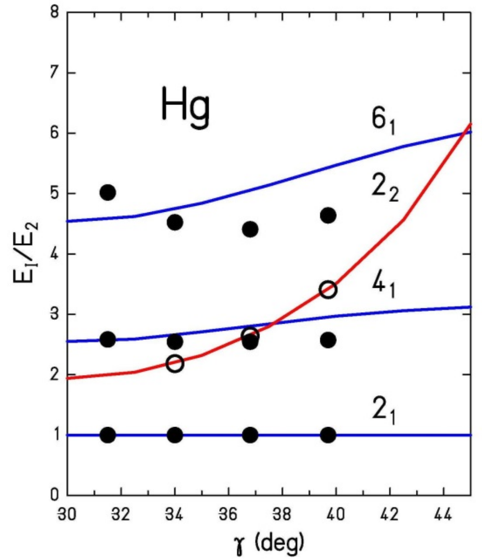

$Q_0$ = 5.8 (5) b
β = 0.227

$Q_0$ = 5.6 (3) b
β = 0.219

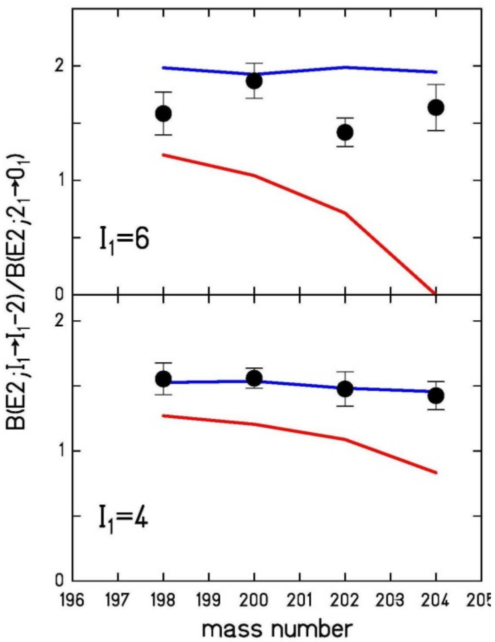
$Q_0$ = 6.3 (4) b
β = 0.239

*R. Kulessa et al.; Phys. Lett B218 (1989), 421*

# Collective properties in $^{198,200,202,204}Hg$




| A | $B(E2;2_1 \rightarrow 0_1)$ |
|---|---|
| 198 | 29 spu |
| 200 | 25 spu |
| 202 | 17 spu |
| 204 | 12 spu |




soft (μ=0.3) **asymmetric rotor model:**

**IBM – O(6) limit:**

$$\frac{B(E2;6_1 \rightarrow 4_1)}{B(E2;2_1 \rightarrow 0_1)} = \frac{5 \cdot (N-2) \cdot (N+6)}{3 \cdot N \cdot (N+4)}$$

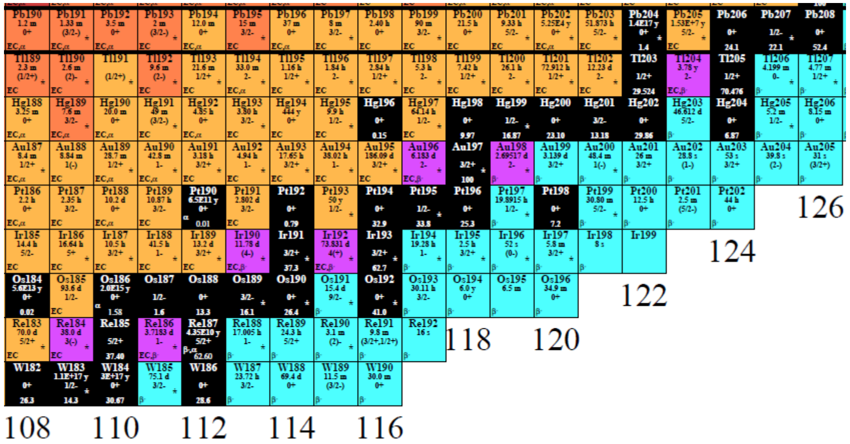
$$\frac{B(E2;4_1 \rightarrow 2_1)}{B(E2;2_1 \rightarrow 0_1)} = \frac{10 \cdot (N-1) \cdot (N+5)}{7 \cdot N \cdot (N+4)}$$

boson number **N = 5** to **2**
for ***Hg-isotopes*** with **A=198** to **204**


*C. Günther et al.; Z. Phys. A301 (1981), 119*
*Y.K. Agarwal et al.; Z. Phys. A320 (1985), 295*

# Parameter of the asymmetric rotor model

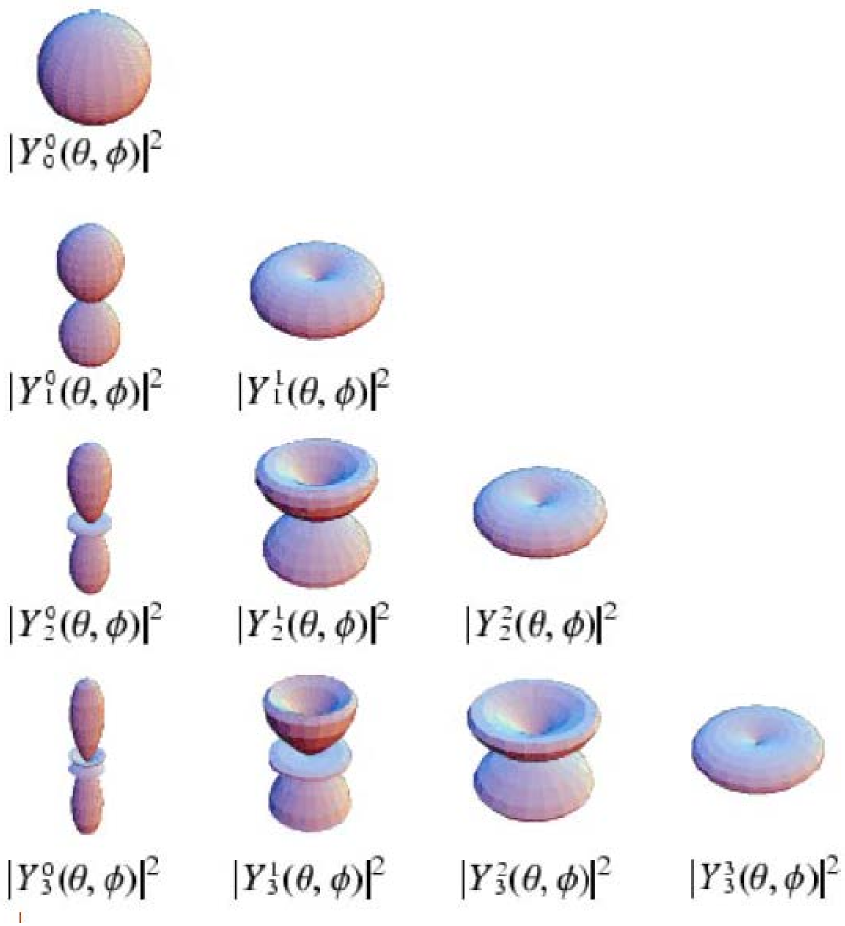
| isotope | β | γ | γ | μ |
|---|---|---|---|---|
| $^{182}$W | 0.274 | 11.4$^0$ | 11.2$^0$ | 0.17 |
| $^{184}$W | 0.258 | 13.8$^0$ | 13.7$^0$ | 0.15 |
| $^{186}$W | 0.223 | 15.9$^0$ | 15.8$^0$ | 0.05 |
| $^{186}$Os | 0.196 | 16.5$^0$ | 16.1$^0$ | 0.26 |
| $^{188}$Os | 0.185 | 19.2$^0$ | 18.8$^0$ | 0.26 |
| $^{190}$Os | 0.184 | 22.3$^0$ | 22.0$^0$ | 0.26 |
| $^{192}$Os | 0.168 | 25.2$^0$ | 25.2$^0$ | 0.10 |
| $^{192}$Pt | 0.146 | - | 32.5$^0$ | 0.35 |
| $^{194}$Pt | 0.134 | - | 32.5$^0$ | 0.35 |
| $^{196}$Pt | 0.135 | - | 32.5$^0$ | 0.37 |
| $^{198}$Hg | 0.106 | 36.3$^0$ | 38.0$^0$ | 0.44 |
| $^{200}$Hg | 0.098 | 39.1$^0$ | 41.0$^0$ | 0.44 |
| $^{202}$Hg | 0.082 | 33.4$^0$ | 34.4$^0$ | 0.35 |
| $^{204}$Hg | 0.068 | 31.5$^0$ | 31.5$^0$ | 0.19 |




$$\frac{E(2_2)}{E(2_1)} = \frac{3+\sqrt{9-8\cdot\sin^2(3\gamma)}}{3-\sqrt{9-8\cdot\sin^2(3\gamma)}}$$

$$B(E2;0_1\rightarrow 2_1) = \frac{5}{16\pi}\cdot Q_0^2 e^2\cdot\frac{1}{2}\cdot\left[1+\frac{3-2\cdot\sin^2(3\gamma)}{\sqrt{9-8\cdot\sin^2(3\gamma)}}\right]$$

$$Q_0 = \frac{3\cdot Z\cdot R_0^2}{\sqrt{5\pi}}\cdot\beta$$




$$Y_{00}(\theta,\phi)=\frac{1}{\sqrt{4\pi}}$$

$$Y_{10}(\theta,\phi)=\frac{1}{2}\cdot\sqrt{\frac{3}{\pi}}\cdot\cos\theta$$

$$Y_{1\pm1}(\theta,\phi)=\mp\frac{1}{2}\cdot\sqrt{\frac{3}{2\pi}}\cdot\sin\theta\cdot e^{\pm i\phi}$$

$$Y_{20}(\theta,\phi)=\sqrt{\frac{5}{16\pi}}\cdot\left(3\cdot\cos^2\theta-1\right)$$

$$Y_{2\pm1}(\theta,\phi)=\mp\sqrt{\frac{15}{8\pi}}\cdot\sin\theta\cdot\cos\theta\cdot e^{\pm i\phi}$$

$$Y_{2\pm2}(\theta,\phi)=\sqrt{\frac{15}{32\pi}}\cdot\sin^2\theta\cdot e^{\pm 2i\phi}$$

$$Y_{30}(\theta,\phi)=\sqrt{\frac{7}{16\pi}}\cdot\left(2\cos^3\theta-3\cos\theta\sin^2\theta\right)$$

$$Y_{3\pm1}(\theta,\phi)=\mp\sqrt{\frac{21}{64\pi}}\cdot\left(4\cos^2\theta\sin\theta-\sin^3\theta\right)\cdot e^{\pm i\phi}$$

$$Y_{3\pm2}(\theta,\phi)=\sqrt{\frac{105}{32\pi}}\cdot\cos\theta\sin^2\theta\cdot e^{(\pm2)i\phi}$$

$$Y_{3\pm3}(\theta,\phi)=\mp\sqrt{\frac{35}{64\pi}}\cdot\sin^3\theta\cdot e^{(\pm3)i\phi}$$