Outline: Nuclear angular momentum

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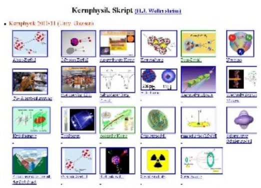
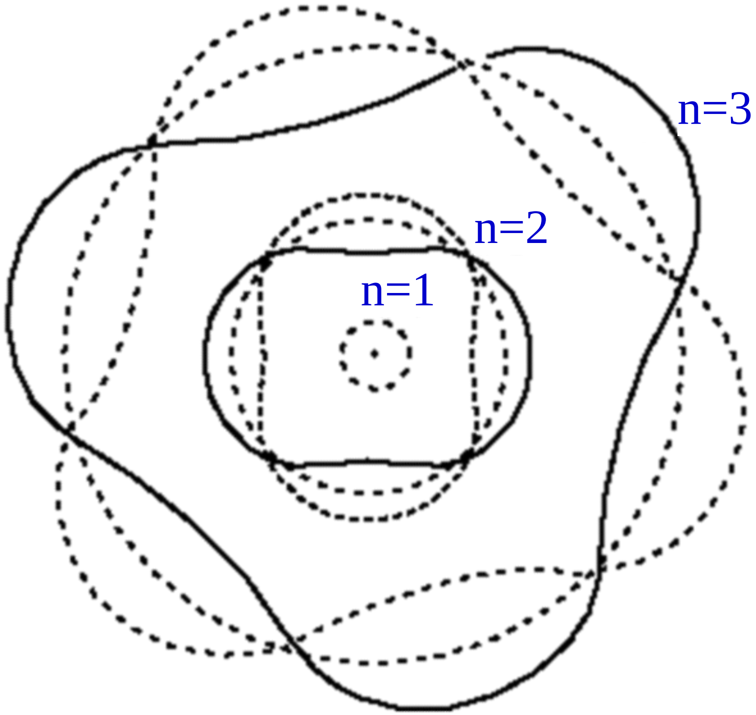
web-page: https://web-docs.gsi.de/~wolle/ and click on



1. nuclear spin quantum number
2. parity
3. magnetic moment
4. magnetic resonance imaging

Nuclear angular momentum

electron orbitals



electrons in an atom

quantum numbers:

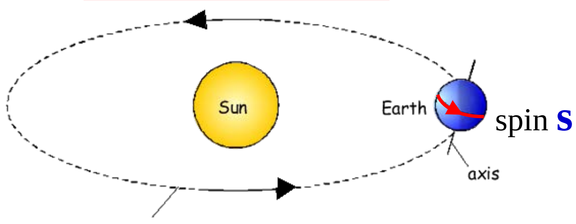
n (principal) **1,2,3,…**

ℓ (orbital angular momentum) **0 → n-1**

m (magnetic) **-ℓ ≤ m ≤ +ℓ**

s (spin) ↑↓ or **+½ħ -½ħ**

classical analogy



sun ≡ nucleus
earth ≡ electron

***protons and neutrons** have* **ℓ** *and* **s**

total angular momentum: $\vec{j} = \vec{\ell} + \vec{s}$

total nuclear spin: $I = \sum j$

electron is structure less and hence can not rotate
***spin s** is a quantum mechanical concept*

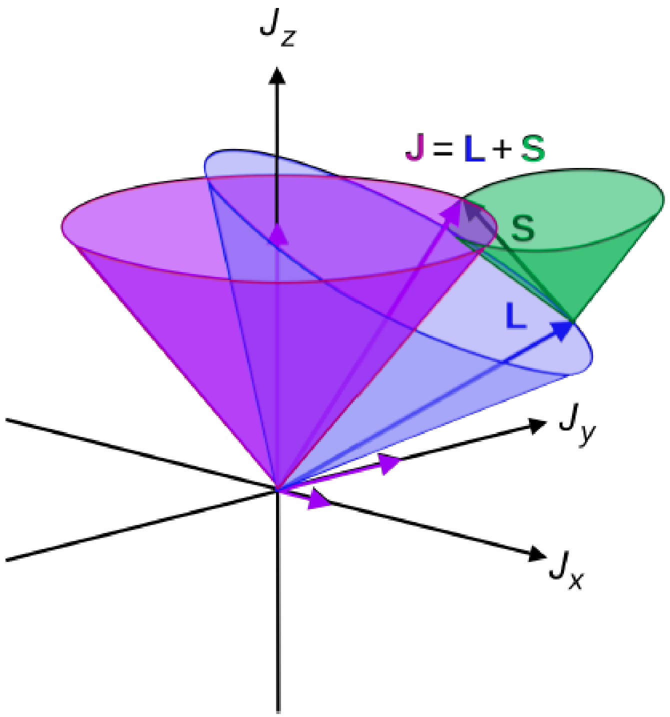
Nuclear spin quantum number

protons and neutrons *have orbital angular momentum* **ℓ** *and spin* **s**

total angular momentum: $\vec{j} = \vec{\ell} + \vec{s}$

total nuclear spin: $I = \sum j$

$$I = |j_1 + j_2 + \cdots j_n|, |j_1 + j_2 + \cdots j_n| - 1, \cdots, |j_1 - j_2 - \cdots - j_n|$$ quantum mechanics



Nuclear spin quantum number

protons and neutrons *have orbital angular momentum* $\boldsymbol{\ell}$ *and spin* $\mathbf{s}$

total angular momentum: $\vec{j} = \vec{\ell} + \vec{s}$

total nuclear spin: $I = \sum j$

$$I = |j_1 + j_2 + \cdots j_n|, |j_1 + j_2 + \cdots j_n| - 1, \cdots, |j_1 - j_2 - \cdots - j_n|$$ quantum mechanics

- ^{1}H = 1 proton, so I = ½
- ^{2}H = 1 proton and 1 neutron, so I = 1 or 0

- For heavier nuclei, it is not immediately evident what the spin should be as there are a multitude of possible values.

mass number	number of protons	number of neutrons	spin (I)	example
even	even	even	0	^{16}O
	odd	odd	integer (1,2,…)	^{2}H
odd	even	odd	half-integer ($^1/_2, ^3/_2, \cdots$)	^{13}C
	odd	even	half-integer ($^1/_2, ^3/_2, \cdots$)	^{15}N

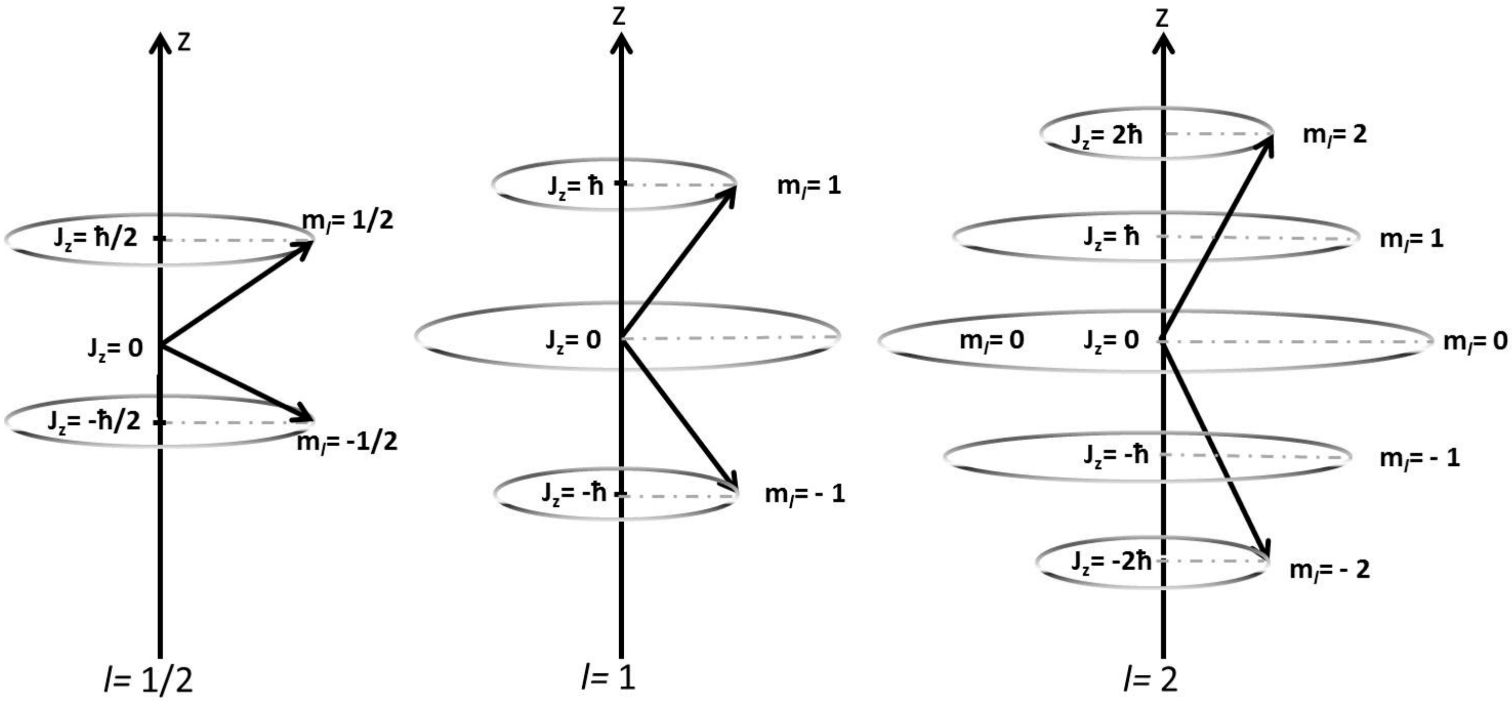
Nuclear spin quantum number

The magnitude is given by

$$L = \hbar\sqrt{I(I+1)}$$

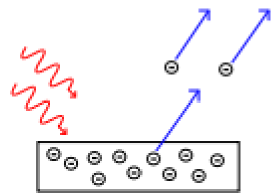
The projection on the z-axis (arbitrarily chosen), takes on discretized values according to m, where

$$m = -I, -I+1, -I+2, \cdots, +I$$



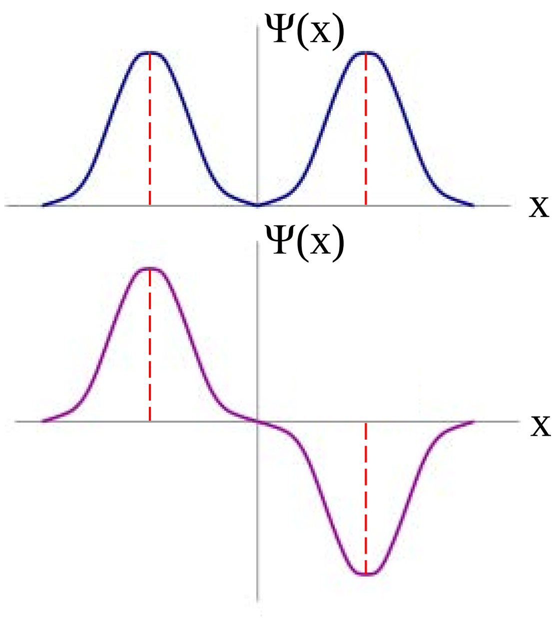
Parity

wave – particle duality:



photoelectric effect

wave function



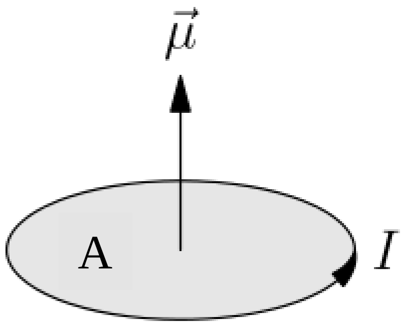
$\Psi(x) = \Psi(-x) \quad \rightarrow \quad$ parity = even (+)

$\ell = 0, 2, 4, \ldots$ even

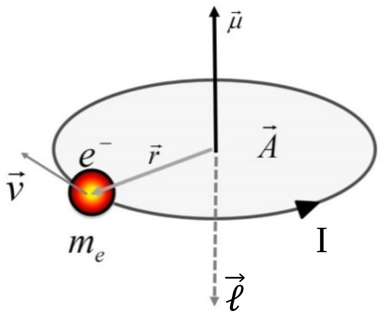
$\Psi(x) = -\Psi(-x) \quad \rightarrow \quad$ parity = odd (-)

$\ell = 1, 3, 5, \ldots$ odd

Magnetic moment



$$\vec{\mu} = I \cdot A$$



$$\ell = m_e v \cdot r \qquad v = \frac{2\pi r}{t} \quad \rightarrow \quad t = \frac{2\pi r}{v}$$

$$I = \frac{e}{t} = \frac{e \cdot v}{2\pi r}$$

$$\mu = \frac{e \cdot v}{2\pi r} \cdot \pi r^2 = \frac{e \cdot v \cdot r}{2}$$

$$\mu = \frac{e \cdot v \cdot r}{2} \cdot \frac{m_e v \cdot r}{m_e v \cdot r} = \frac{e \cdot \ell}{2m_e} \qquad \mu_{Bohr} = \frac{e \cdot \hbar}{2m_e}$$

$$\mu_\ell = \mu_B \cdot \frac{\ell}{\hbar}$$

electron orbital magnetic moment

$$\mu_\ell = \mu_N \cdot \frac{\ell}{\hbar}$$

proton orbital magnetic moment $\quad \mu_N = \frac{e \cdot \hbar}{2m_p}$

Magnetic moment

$$\mu_\ell = \mu_B \cdot \frac{\ell}{\hbar}$$

electron orbital magnetic moment

$$\mu_s = -2.0023 \cdot \mu_B \cdot \frac{s}{\hbar}$$

electron spin magnetic moment (Dirac equation)

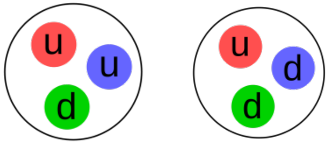
$$\mu_s = +5.585691 \cdot \mu_N \cdot \frac{s}{\hbar}$$

proton spin magnetic moment

$$\mu_s = -3.826084 \cdot \mu_N \cdot \frac{s}{\hbar}$$

neutron spin magnetic moment

Why has a neutron a magnetic moment when it is uncharged?



proton $+1e$ neutron $0e$

u-quark: $+{}^{2}/_{3}\, e$
d-quark: $-{}^{1}/_{3}\, e$

neutrons and protons are not elementary particles
internal structure: they have charges.

$$\mu_s({}^{2}_{1}H) = (5.59 - 3.83) \cdot \mu_N \cdot \frac{1}{2} = 0.87980\mu_N$$

$$= 0.8574 \cdot \mu_N$$ (experiment)

Quark bag model

	Charge	*Mass m_q*	*Spin*	*mag. Moment*
u-quark	+ 2/3	~ 1/3 M_p	1/2	$2\cdot\left(\frac{2}{3}\right)\cdot\frac{e\cdot\hbar}{2\cdot m_q\cdot c}\cdot s=4\cdot\mu_K\cdot s$
d-quark	- 1/3	~ 1/3 M_p	1/2	$2\cdot\left(-\frac{1}{3}\right)\cdot\frac{e\cdot\hbar}{2\cdot m_q\cdot c}\cdot s=-2\cdot\mu_K\cdot s$

general coupling rule:

$$g(j_1\times j_2;J)=\frac{1}{2}(g_1+g_2)+\frac{j_1(j_1+1)-j_2(j_2+1)}{2\cdot J\cdot(J+1)}\cdot(g_1-g_2)$$

Proton (uud) :

$$g(1/2\times 1/2;1)=\frac{1}{2}(4+4)+\frac{1/2\cdot 3/2-1/2\cdot 3/2}{2\cdot 1\cdot 2}\cdot(4-4)=4$$

$$g(1\times 1/2;1/2)=\frac{1}{2}(4-2)+\frac{2-3/4}{3/2}\cdot(4+2)=\boxed{+6}$$

Neutron (ddu) :

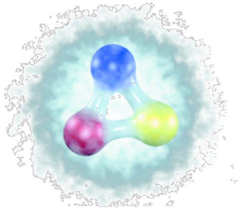
$$g(1/2\times 1/2;1)=\frac{1}{2}(-2-2)+\frac{1/2\cdot 3/2-1/2\cdot 3/2}{2\cdot 1\cdot 2}\cdot(-2+2)=-2$$

$$g(1\times 1/2;1/2)=\frac{1}{2}(-2+4)+\frac{2-3/4}{3/2}\cdot(-2-4)=\boxed{-4}$$

$g_{Dirac}=2$ prediction of the Dirac-theory for the gyromagnetic factor for spin ½-particles

$g_s^{proton}=+5.58$ ⇒ big deviation from g = 2 ⇔ no fundamental particle

$g_s^{neutron}=-3.82$



Application: Magnetic resonance imaging (MRI)

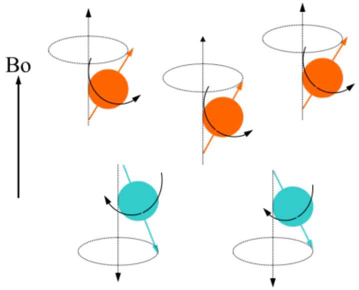
$\mu_s = +5.585691 \cdot \mu_N \cdot \frac{s}{\hbar}$ proton spin magnetic moment

$\mu_I = \gamma \cdot I$

gyromagnetic ratio $\gamma = g \cdot \frac{\mu_N}{\hbar} = g \cdot 7.62 \cdot 10^6 \quad [T^{-1}s^{-1}]$

proton g-factor: +5.585691, spin I: ½ ħ

proton in magnetic field



Parallel (low energy)

Anti-Parallel (high energy)

energy difference between states

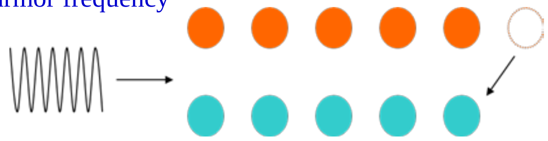
$\Delta E = h \cdot \nu$

$\Delta E = 2 \cdot \mu_I \cdot B_0$

$\nu = {}^{\gamma}/_{2\pi} \cdot B_0$ Larmor frequency

${}^{\gamma}/_{2\pi} = 42.57 \quad [{}^{MHz}/_{T}]$ for proton

Larmor frequency

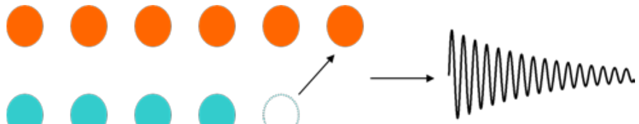


low energy

high energy

low energy

high energy



relaxation (tissue dependent)