Outline:

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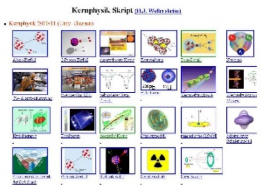
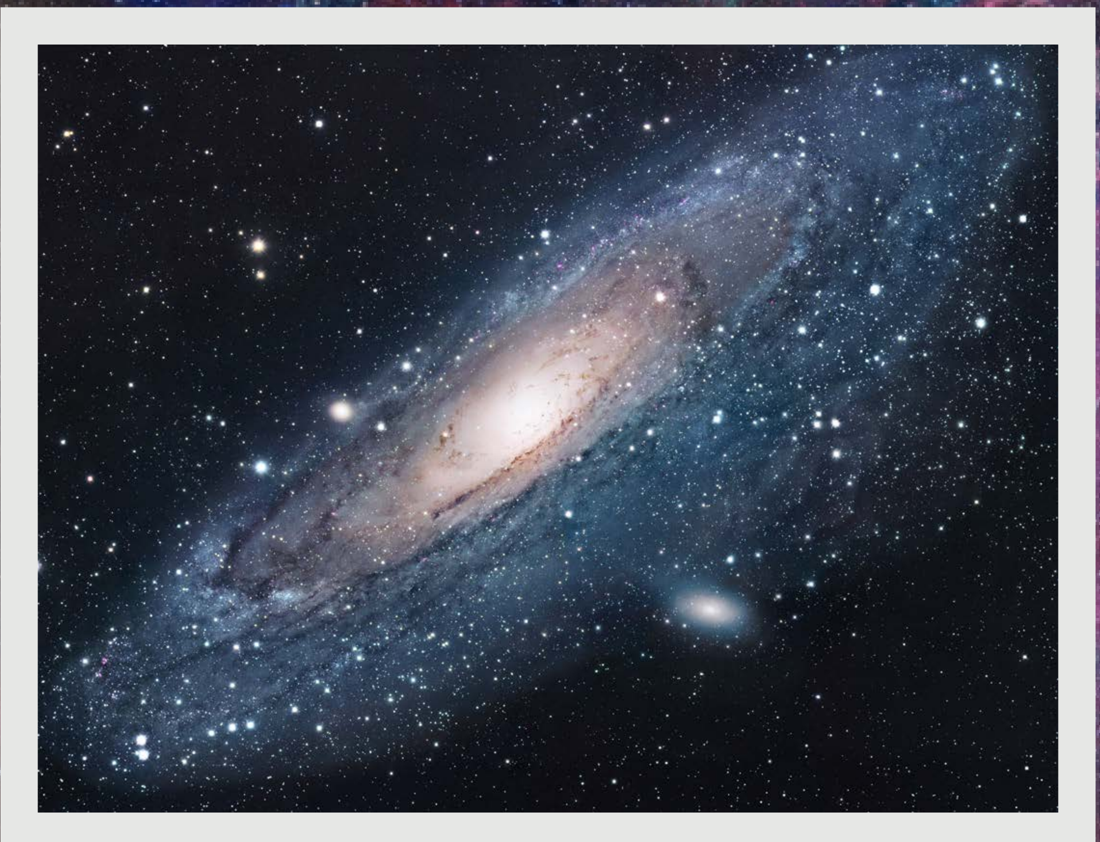
web-page: https://web-docs.gsi.de/~wolle/ and click on 

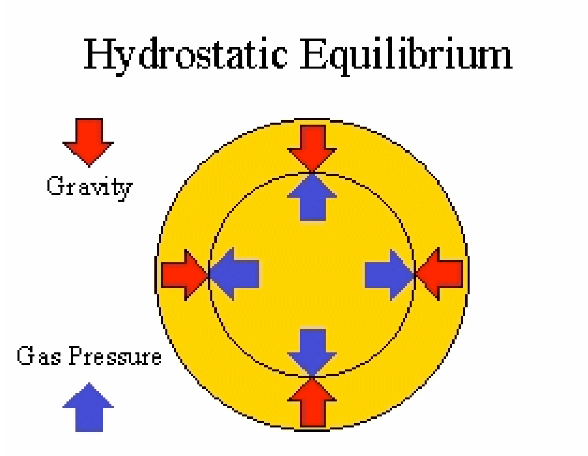
1. hydrostatic equilibrium
2. ideal gas assumption
3. virial theorem
4. timescales

Stars are the building blocks of our Universe



Hydrostatic Equilibrium: Stars as self-regulating system

- Energy is generated in the star's hot core, then carried outward to the cooler surface.
- Inside a star, the inward force of gravity is balanced by the outward force of pressure.
- The star is stabilized (i.e. nuclear reactions are kept under control) by a pressure-temperature thermostat.



Hydrostatic Equilibrium: Stars as self-regulating system

Self-Regulation in Stars

Suppose the fusion rate increases slightly.
Then,
(1) Temperature increases.
(2) Pressure increases.
(3) Core expands.
(4) Density and temperature decrease.
(5) Fusion rate decreases.

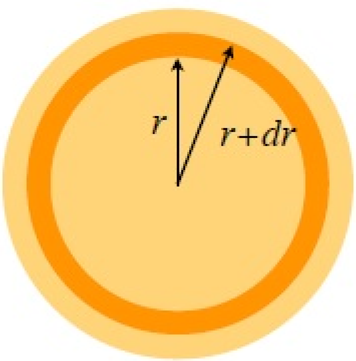
So there's a feedback mechanism which prevents the fusion rate from skyrocketing upward. We can reverse this argument as well …
Now suppose that there was no source of energy in stars (e.g., no nuclear reactions)

Core Collapse in a Self-Gravitating System

- Suppose that there was no energy generation in the core. The pressure would still be high, so the core would be hotter than the envelope.
- Energy would escape (via radiation, convection…) and so the core would shrink a bit under the gravity
- That would make it even hotter, and then even more energy would escape; and so on, in a feedback loop
- ➡ **Core collapse!** Unless an energy source is present to compensate for the escaping energy.
- In stars, nuclear reactions play this role. In star clusters, hard binaries do.

Conservation of mass

- If ***m*** is the mass interior to radius r, then ***m***, ***r*** and ***ρ*** are not independent, because ***m(r)*** is determined by ***ρ(r).***
- Consider a thin shell inside the star, radius ***r*** and thickness ***dr***



Volume is $dV = 4\pi r^2 dr$, so mass of shell is

$$dm = 4\pi r^2 dr \cdot \rho(r)$$

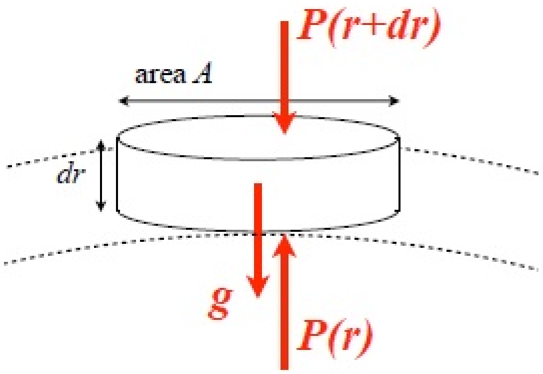
or

$$\frac{dm}{dr} = 4\pi r^2 \cdot \rho(r)$$

the equation of **mass conservation**

Hydrostatic equilibrium

- Consider a small parcel of gas at a distance ***r*** from the centre of the star, with density $\rho(r)$, area ***A*** and thickness ***dr***.
- Outward force: pressure on bottom face $P(r) \cdot A$



- Inward force: pressure on top face, plus gravity due to material interior to ***r***:

$$P(r) \cdot A = P(r + dr) \cdot A + \frac{G \cdot m(r) \cdot dm}{r^2}$$

$$= P(r + dr) \cdot A + \frac{G \cdot m(r) \cdot \rho \cdot A \cdot dr}{r^2}$$

$$\frac{P(r + dr) - P(r)}{dr} \cdot A \cdot dr = -\frac{G \cdot m(r)}{r^2} \cdot \rho(r) \cdot A \cdot dr$$

$$\boxed{\frac{dP}{dr} = -\frac{G \cdot m}{r^2} \cdot \rho}$$

the equation of **hydrostatic equilibrium**

alternate form:
$$\frac{dP}{dm} = \frac{dP}{dr} \cdot \frac{dr}{dm} = -\frac{G \cdot m}{r^2} \cdot \rho \cdot \frac{1}{4\pi \cdot r^2 \rho} = -\frac{G \cdot m}{4\pi r^4}$$

Estimate for central pressure

- We can use hydrostatic equilibrium to estimate P_C: we approximate the pressure gradient as a constant

$$\frac{dP}{dr} \sim -\frac{\Delta P}{\Delta R} = \frac{P_c}{R} = \frac{G \cdot M}{R^2} \cdot \rho$$

Now assume the star has constant density, so

$$P_C = \bar{P} = \frac{M}{V} \sim \frac{M}{\frac{4}{3} \cdot \pi \cdot R^3}$$

then so

$$P_C \sim \frac{3 \cdot G \cdot M^2}{4 \cdot \pi \cdot R^4}$$

For the Sun, we estimate $P_C \sim 3 \cdot 10^{14}\ Nm^{-2} = 3 \cdot 10^9\ atm$

($R = 7 \cdot 10^8$ m, $M = 2 \cdot 10^{30}$ kg, $G = 6{,}67 \cdot 10^{-11}$ N m^2 kg^{-2})

Equation of state in stars

Interior of a star contains a mixture of ions, electrons, and radiation (photons). For most stars (exception very low mass stars and stellar remnants) the ions and electrons can be treated as an ideal gas and quantum effects can be neglected.

Total pressure:

$$\begin{aligned} P &= P_I + P_e + P_r \\ &= P_{gas} + P_r \end{aligned}$$

- P_I is the pressure of the ions
- P_e is the electron pressure
- P_r is the radiation pressure

Gas pressure

The equation of state for an ideal gas is:

$$P_{gas} = n \cdot k \cdot T$$

n is the number of particles per unit volume; $n = n_I + n_e$, where n_I and n_e are the number densities of ions and electrons

In terms of the mass density ρ:

$$P_{gas} = \frac{\rho}{\mu \cdot m_H} \cdot k \cdot T$$

…where m_H is the mass of hydrogen and μ is the average mass of particles in units of m_H. Define the **ideal gas constant**:

$$R \equiv \frac{k}{m_H} \quad \Rightarrow \quad P_{gas} = \frac{R}{\mu} \cdot \rho \cdot T$$

Determining mean molecular weight μ

μ will depend upon the composition of the gas and the state of ionization. For example:

- Neutral hydrogen: $\mu = 1$
- Fully ionized hydrogen: $\mu = 0.5$

In the central regions of stars, OK to assume that all the elements are fully ionized.

Denote abundances of different elements per unit mass by:

- **X** hydrogen - mass m_H, one electron
- **Y** helium - mass $4m_H$, two electrons
- **Z** the rest, `metals', average mass $A \cdot m_H$, approximately (A / 2) electrons per nucleus

Determining mean molecular weight μ

If the density of the plasma is ρ, then add up number densities of hydrogen, helium, and metal nuclei, plus electrons from each species:

	H	**He**	**metals**
number density of nuclei	$\frac{X\rho}{m_H}$	$\frac{Y\rho}{4 \cdot m_H}$	$\frac{Z \cdot \rho}{A \cdot m_H}$
number density of electrons	$\frac{X\rho}{m_H}$	$\frac{2 \cdot Y\rho}{4 \cdot m_H}$	$\approx \frac{A}{2} \cdot \frac{Z \cdot \rho}{A \cdot m_H}$

$$n = \frac{\rho}{m_H}\left[2 \cdot X + \frac{3}{4} \cdot Y + \frac{1}{2} Z\right] = \frac{\rho}{\mu \cdot m_H}$$

… assuming that A >> 1

$$\boxed{\mu^{-1} = 2 \cdot X + \frac{3}{4} \cdot Y + 2 \cdot X}$$

Mean molecular weight μ

Consider the mean mass $\bar{m}$ per particle

$$\bar{m} = \frac{\sum_j n_{j,I} \cdot m_{jI} + n_e m_e}{\sum_j n_{jI} + n_e} \approx \frac{\sum_j n_{j,I} \cdot m_{j,I}}{\sum_j n_{j,I} + n_e}$$

where $n_{j,I}$ is the ion number density of ion j, $m_{j,I}$ is its mass, and n_e and m_e are the numbers and mass of the electron (and then we ignore the electron mass).

- The mass of the jth ion is approximately its number of protons and neutrons (A_j) times the amu, or $m_{j,I} = A_j \cdot m_u$.
- So then we define

$$\mu = \frac{\bar{m}}{m_u} = \frac{\sum_j n_{j,I} \cdot A_j}{\sum_j n_{j,I} + n_e}$$

This can be interpreted as the average mass per particle (ion, electron, etc.) in units of the amu.

Note that the total particle number density in the gas is

$$n = n_e + n_I = n_e + \sum_j n_{j,I} = \sum_j (1 + Z_j) \cdot n_{j,I}$$

since one ionized atom contributes 1 nucleus plus Z_j electrons. (Z_j is charge of each nucleus)

Mean molecular weight μ

REMINDER

Imagine a star where 92% of all particles are hydrogen nuclei and 8% of them are helium nuclei. What are the mass fractions of hydrogen and helium?

$$
\begin{aligned}
92 &= \frac{\rho x_{\mathrm{H}}}{m_{\mathrm{u}}} \\
8 &= \frac{\rho x_{\mathrm{He}}}{4 m_{\mathrm{u}}} \\
x_{\mathrm{H}} &= 92 m_{\mathrm{u}}/\rho \\
x_{\mathrm{He}} &= 32 m_{\mathrm{u}}/\rho \\
92 m_{\mathrm{u}}/\rho + 32 m_{\mathrm{u}}/\rho &= 124 m_{\mathrm{u}}/\rho = 1 \\
m_{\mathrm{u}}/\rho &= 1/124 \\
x_{\mathrm{H}} &= 92/124 = 0.7419 \\
x_{\mathrm{He}} &= 32/124 = 0.2581.
\end{aligned}
$$

REMINDER

Mean molecular weight μ

So using this, we now have

$$\mu = \frac{\sum_i \frac{\rho}{m_u} x_i}{\sum_i \frac{\rho x_i}{m_u A_i} + n_e},$$

or

$$\mu = \frac{\sum_i \frac{\rho}{m_u} x_i}{\sum_i \frac{\rho x_i}{m_u A_i}(1+Z_i)}.$$

One cleaner way of writing this is

$$\mu^{-1} = \frac{\sum_i x_i/A_i(1+Z_i)}{\sum_i x_i} = \sum_i \frac{x_i}{A_i}(1+Z_i).$$

For example, for a neutral gas ($Z = 0$) we have

$$\mu^{-1} = \sum_i \frac{x_i}{A_i} \approx \left(X + \frac{Y}{4} + \frac{Z}{\overline{A_i}}\right)^{-1},$$

where it is standard to write mass fractions X for hydrogen, Y for helium, and Z for everything else (metals), where $X + Y + Z = 1$.

Mean molecular weight μ

For a fully ionized gas

$$\mu^{-1} \approx \sum_i \frac{x_i}{A_i}(1 + Z_i) \approx 2X + \frac{3}{4}Y + \frac{1}{2}Z,$$

or

$$\mu \approx \frac{4}{3 + 5X - Z},$$

where for metals we usually approximate $(1 + Z_i)/A_i \approx 1/2$ (roughly equal number of protons and neutrons). We've eliminated Y in this expression through $Y = 1 - X - Z$.

Mean molecular weight μ

Compute the mean molecular weight for (1) the ionized solar photosphere, where we have 90% hydrogen, 9% helium, and 1% heavy elements by mass, (2) the ionized solar interior where 71% hydrogen, 27% helium, and 2% heavy elements by mass, (3) completely ionized hydrogen, (4) completely ionized helium, and finally (5) neutral gas at the solar interior abundance.

Answer: (1) For the photosphere we can write

$$\mu^{-1} = 0.9\frac{2}{1} + 0.09\frac{3}{4} + 0.01\frac{1}{2} = 1.8725,$$

or $\mu \approx 0.53$.
(2) For the interior we can write

$$\mu^{-1} = 0.71\frac{2}{1} + 0.27\frac{3}{4} + 0.02\frac{1}{2} = 1.63,$$

or $\mu \approx 0.61$.
(3) For hydrogen, we will take $X = Z = A = 1$, and find then that $1/\mu = 2$.
(4) For helium, $X = Z = 0$ and $Y = 1$, so $\mu = 4/3$.
(5) For a neutral gas, we have

$$\mu^{-1} = 0.71 + 0.27\frac{1}{4} + 0.02\frac{1}{15.5} = 0.779,$$

or $\mu \approx 1.28$.

Compute an expression for μ_e in the deep stellar interior as a function only of X. Ignore metals.

Answer: Fully ionized case. We can write

$$
\begin{aligned}
\mu_e &\approx \left(\frac{1}{1}X + \frac{2}{4}Y\right)^{-1} \\
&= \left(X + \frac{1}{2}(1-X)\right)^{-1} \\
&= \left(\frac{X+1}{2}\right)^{-1} = \frac{2}{1+X}.
\end{aligned}
$$

This should make sense. For a full H gas, the mean mass of particles per number of electrons (1/1) is 1. For a He gas ($X = 0$), we have a mass of 4 divided by 2 electrons, or $\mu_e = 2$.

Some Order-of-Magnitude

Let's see if we can estimate roughly the conditions in the Solar core.

- **Pressure** P = F / A:

$$F \approx G \cdot M_\odot^2 / R_\odot^2$$
$$A \approx 4\pi R_\odot^2$$
$$P \approx G \cdot M_\odot^2 / 4\pi R_\odot^4$$

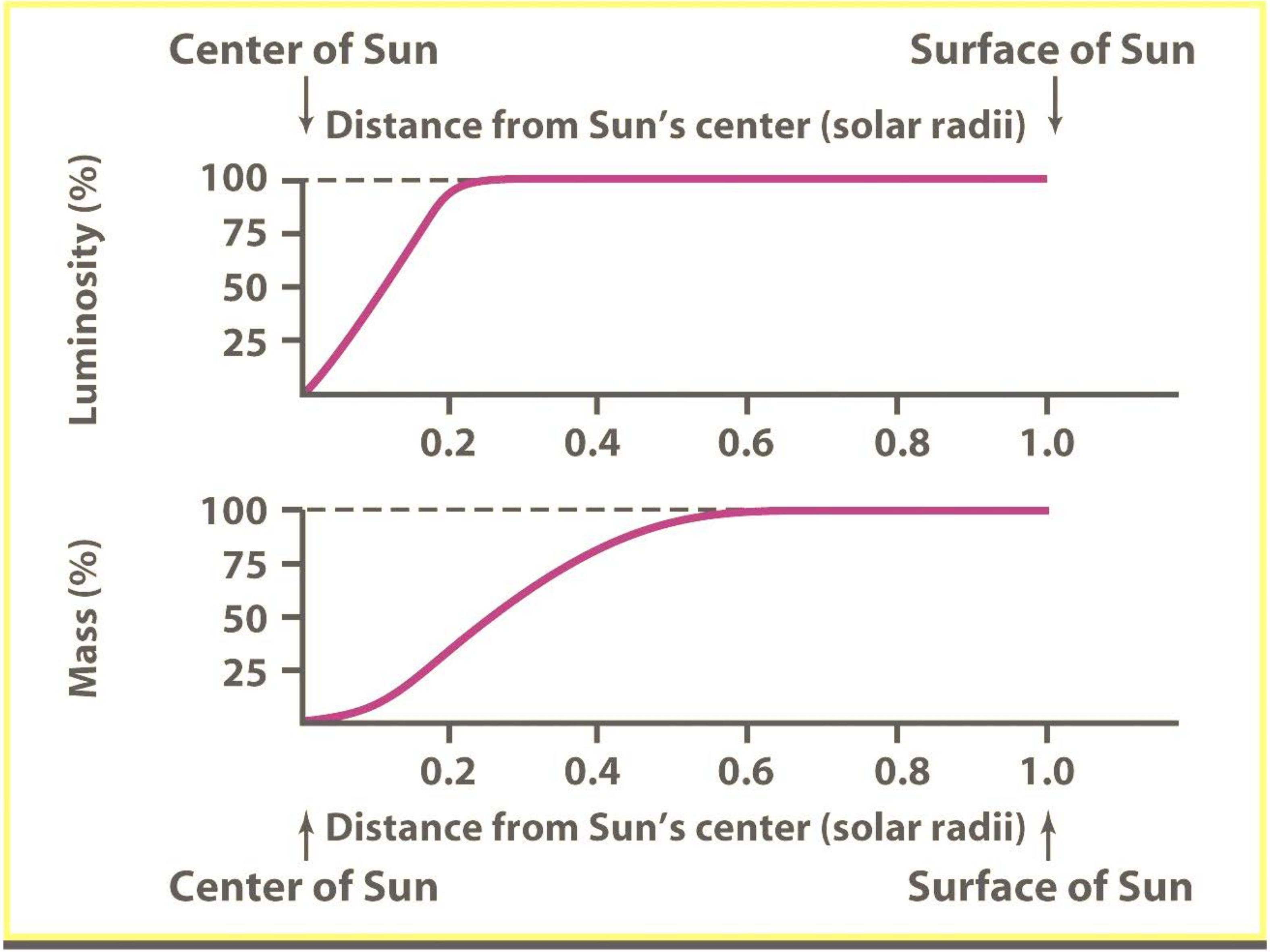
$(M_\odot \approx 2 \cdot 10^{30} kg, R_\odot \approx 7 \cdot 10^8 m, G \approx 6.67 \cdot 10^{-11} Nm^2 kg^{-2})$

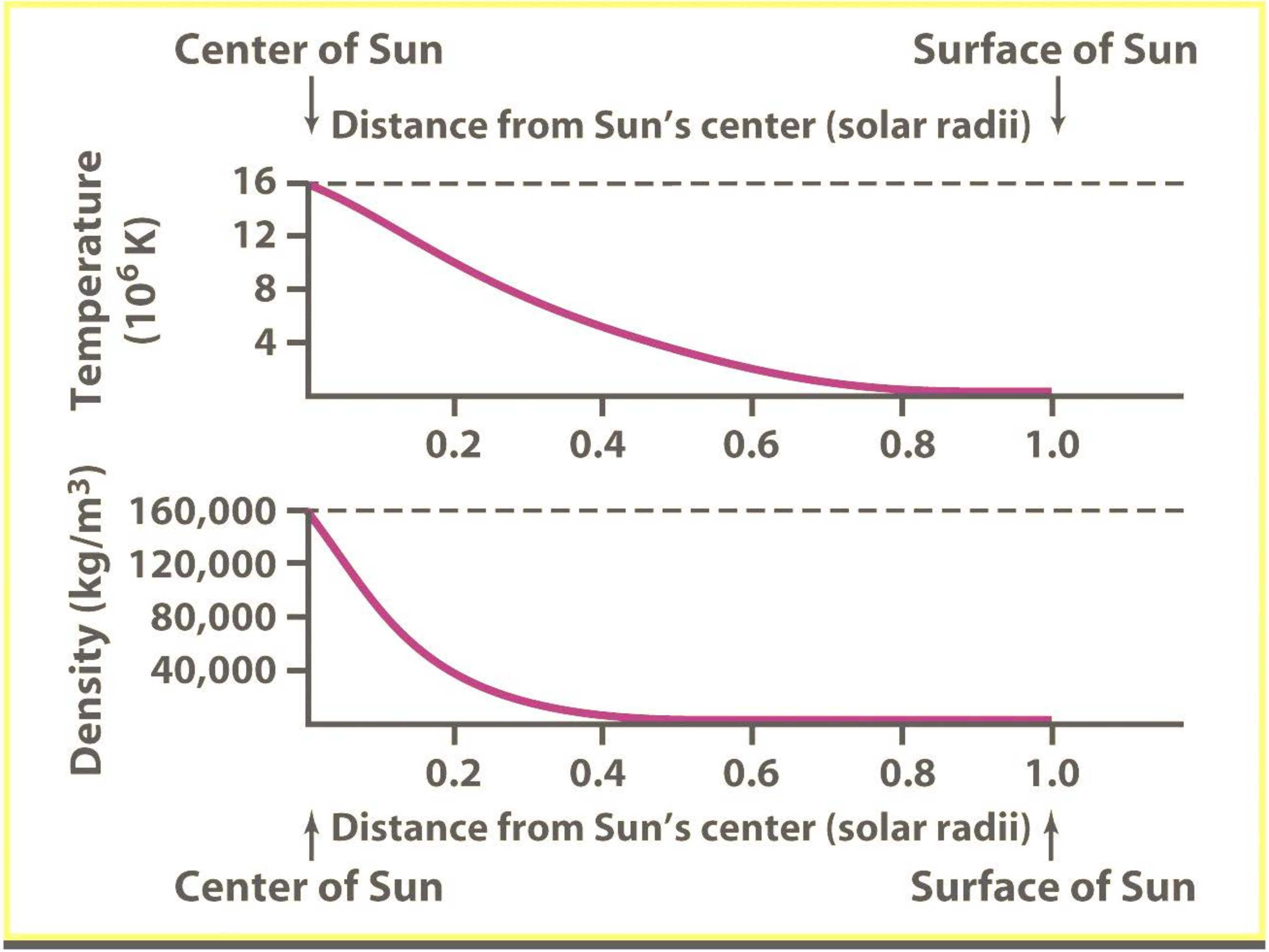
Thus: $P_{est} \sim 10^{14}$ N/m^2 - and surely an underestimate
True value: $P_c \approx 2 \cdot 10^{16}$ N/m^2

Now the **temperature:** $\boldsymbol{3/2 \cdot k \cdot T \approx G \cdot m_p \cdot M_\odot / R}$
$(k \approx 1.4 \cdot\ 10^{-23} J/K \approx 1.7 \cdot 10^{-27} kg)$

Thus: $T_{est} \approx 1.6 \cdot 10^7 K$
True value: $T_C \approx 1.57 \cdot 10^7 K$ – not bad!

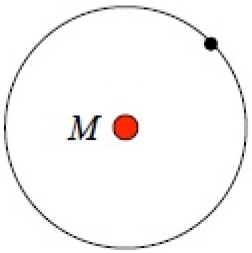
(R = $7 \cdot 10^8$ m, M = $2 \cdot 10^{30}$ kg, $m_H = 1.67 \cdot 10^{-27}$ kg, G = $6{,}67 \cdot 10^{-11}$ Nm^{2}kg^{-2})


Center of Sun
Surface of Sun
Distance from Sun's center (solar radii)
Luminosity (%)
100
75
50
25
0.2
0.4
0.6
0.8
1.0
Mass (%)
100
75
50
25
0.2
0.4
0.6
0.8
1.0
Distance from Sun's center (solar radii)
Center of Sun
Surface of Sun



Center of Sun
Surface of Sun
Distance from Sun's center (solar radii)
Temperature (10^6 K)
16
12
8
4
0.2
0.4
0.6
0.8
1.0
Density (kg/m^3)
160,000
120,000
80,000
40,000
0.2
0.4
0.6
0.8
1.0
Distance from Sun's center (solar radii)
Center of Sun
Surface of Sun

- Gravity has a very important property which relates the gravitational energy of a star to its thermal energy.
- Consider a particle in a circular orbit of radius ***r*** around a mass ***M***



- Potential energy of particle is

$$\Omega = -\frac{G \cdot M \cdot m}{r}$$

- Velocity of particle is $v = \sqrt{\frac{G \cdot M}{r}}$ *(Kepler)*
- So kinetic energy is $K = \frac{1}{2} m \cdot v^2 = \frac{G \cdot M \cdot m}{r}$

  i.e. 2K=-Ω or 2K+Ω=0

- Total energy $E = K + \Omega$

$$= -\frac{\Omega}{2} + \Omega = \frac{\Omega}{2} < 0$$

- Consequence: when something loses energy in gravity it **speeds up**!

The virial theorem

- The virial theorem turns out to be true for a wide variety of systems from clusters of galaxies to an ideal gas; thus for a star we also have

$$\Omega + 2 \cdot U = 0$$

- where U is the total internal (thermal) energy of the star and Ω is the total gravitational energy.
- So a decrease in total energy E leads to a decrease in Ω but an increase in U and hence T, i.e. when a star loses energy, it **heats up**
- **Fundamental principle:** stars have a negative heat capacity: they heat up when their total energy decreases.
- This fact governs the fate of stars

Timescales

There are three important timescales in the life of stars:

- dynamical timescale – *the time scale on which a star would expand or contract if the balance between pressure gradients and gravity was suddenly disrupted*
- thermal timescale – *how long a star would take to radiate away its thermal energy if reactions stopped*
- nuclear timescale – *how long a star would take to exhaust its nuclear fuel at current rate*

- **Dynamical timescale:**
  the timescale on which a star would expand or contract if it were not in equilibrium, also called the **free-fall timescale**

$$\tau_{dyn} \equiv \frac{characteristic\ radius}{characteristic\ velocity} = \frac{R}{v_{esc}}$$

Escape velocity from the surface of the star:

$$v_{esc} = \sqrt{\frac{2 \cdot G \cdot M}{R}} = 620\ \mathrm{km/s}$$

$$\tau_{dyn} = \sqrt{\frac{R^3}{2 \cdot G \cdot M}}$$

For the Sun (R = $7 \cdot 10^8$ m, M = $2 \cdot 10^{30}$ kg, G = $6{,}67 \cdot 10^{-11}$ m^3s^{-2}), $\tau_{dyn} \cong 1100\ s$

Timescales

Thermal timescale:

- the timescale for the star to radiate away its energy if nuclear reactions were switched off: also called **Kelvin-Helmholtz timescale**
- Total gravitational energy available

$$E_{grav} \sim \frac{G \cdot M^2}{R}$$

- If the star radiates energy at $L\ (J/s)$, then it can keep up this rate for

$$\tau_{th} \sim \frac{E_{grav}}{L} \sim \frac{G \cdot M^2}{R \cdot L}$$

For the Sun, $\tau_{th} \cong 3 \cdot 10^7\ y\ \ll$ age of Earth

Timescales

Nuclear timescale: times to exhaust nuclear fuel at current rate.

$$\tau_{nuc} \sim \frac{\eta \cdot M_C \cdot c^2}{L}$$

where η is an efficiency factor for nuclear fusion: η ~ 0.7% and M_C is the mass of the core.

For the Sun, $\tau_{th} \sim 10^{10}\ y$

Binding energy

E.g. hydrogen burning: 4 H $\longrightarrow$ He

$$m_{\mathrm{H}} = 1.0081u, m_{\mathrm{He}} = 4.0039u$$

so

$$\begin{aligned} E &= (4 \times 1.0081 - 4.0039)c^2 = 2.85 \times 10^{-2}c^2 \\ &= 26.7 \text{ MeV per He nucleus} \end{aligned}$$

Fraction of rest-mass energy liberated:

$$\varepsilon = 2.85 \times 10^{-2}/(4 \times 1.0081) = 0.007$$

The transformation of H into He liberates 0.7% of the rest-mass of the system in the form of energy

Timescales

- For stars,

$$\tau_{dyn} \ll \tau_{th} \ll \tau_{nuc}$$

- τ_{dyn} = timescale of collapsing star, e.g. supernova

- τ_{th} = timescale of star before nuclear fusion starts, e.g. pre-main sequence lifetime

- τ_{nuc} = timescale of star during nuclear fusion, e.g. main-sequence lifetime

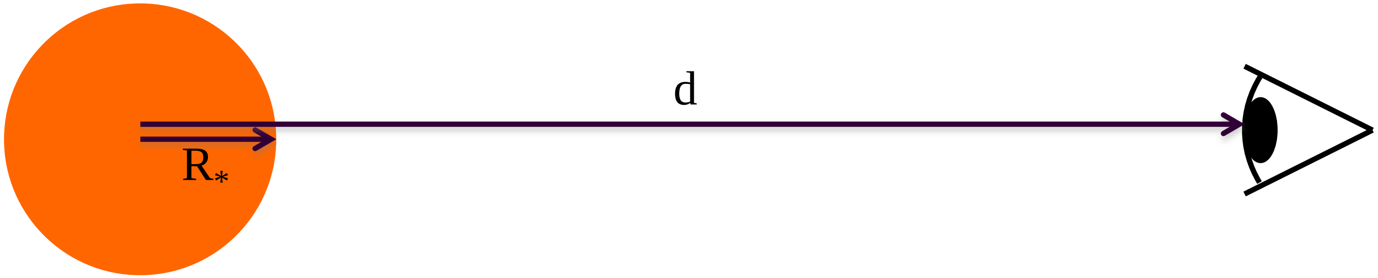
Timescales

- Most stars, most of the time, are in hydrostatic and thermal equilibrium, with slow changes in structure and composition occurring on the (long) timescale τ_{nuc} as fusion occurs

- If something happens to a star faster than one of these timescales, then it will NOT be in equilibrium.

  e.g. sudden addition of energy (nearby supernova?), sudden loss of mass (binary interactions)

Stars show spectra very close to black-body radiation



J. Stefan & L. Boltzmann

flux at the surface:

$$F = \sigma_{SB} \cdot T_*^4$$ with $\sigma_{SB} = 5.67 \cdot 10^{-8}$ W m^{-2} K^{-4}

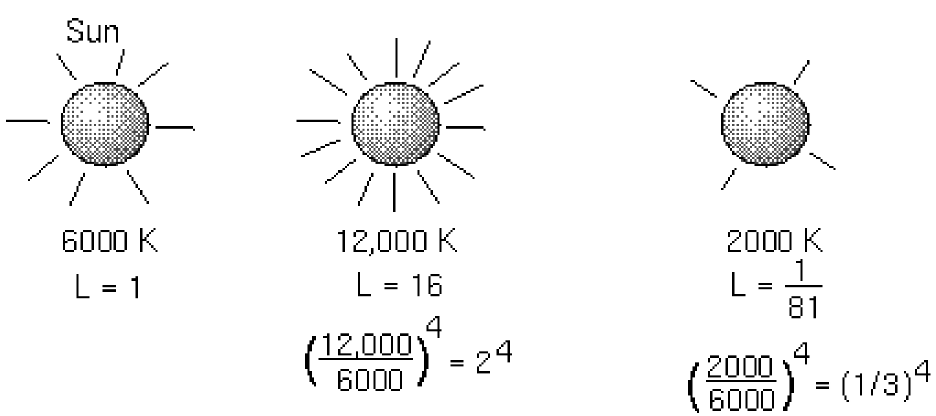
measured flux:

$$F = \left(\frac{R_*}{d}\right)^2 \cdot \sigma_{SB} \cdot T_*^4$$

luminosity (Stefan-Boltzmann law) is the flux multiplied by entire spherical surface:

$$L = 4\pi \cdot R_*^2 \cdot \sigma_{SB} \cdot T_*^4$$

Luminosity is proportional to *fourth* power of temperature.



Hertzsprung-Russell diagram

M, R, L and T_e do not vary independently.

Two major relationships – L with T

– L with M

The first is known as the *Hertzsprung-Russell* (HR) diagram or the *colour-magnitude* diagram.